Math 1B Quiz #10

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Name: ______________________________________

1. (3 pts) Show that $f(t) = Ce^t - t - 1$ is a one-parameter family of solutions to the differential equation
$$\frac{df}{dt} = f(t) + t.$$
If $f(0) = 1$, what is $f(t)$?

We compute and compare the right- and left-hand sides of the equation:

$$\begin{aligned} \frac{df}{dt} &= Ce^t - 1 \\ f + t &= Ce^t - t - 1 + t \\ &= Ce^t - 1 \qquad \checkmark \end{aligned}$$

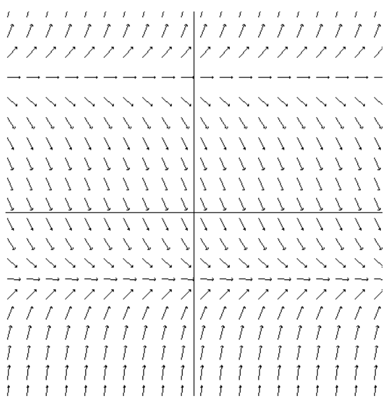
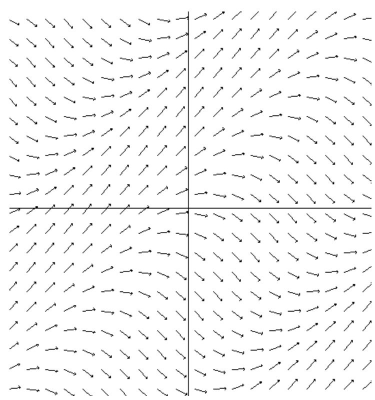
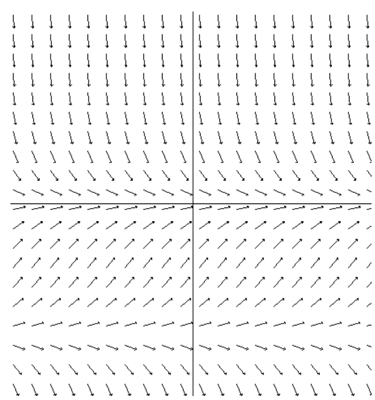
Now we solve the initial-value problem:

$$\begin{aligned} 1 &= f(0) \\ &= Ce^0 - 0 - 1 \\ &= C - 1 \\ C &= 2 \\ f(t) &= 2e^t - t - 1 \end{aligned}$$

2. (3 pts) Which of the three direction fields corresponds to the differential equation

$$\frac{df}{dt} = (f+1)(f-2)?$$

Each is graphed with f and t ranging from -3 to 3. Based on the graph (or any other way), specify which values of f are equilibrium values. Which of these equilibria are stable and which are unstable as $t \to \infty$? Sketch the solution to this differential equation that satisfies the initial value condition that $f = 0$ when $t = 0$.

The first picture is the graph of the differential equation. Upon inspecting the differential equation, we see that the right-hand side $(f+1)(f-2)$ does not depend on t — so the picture must be invariant under left-and-right translations — and has zeros at $f = 1$ and $f = -2$ — these become equilibrium values. Checking a few points (e.g. $f = 0$, $f = 3$, and $f = -1$) gives the overall direction of the slope field in each section of the graph.

An equilibrium is "stable" when: if $f(t)$ is close to the equilibrium value for some t, then $f(t)$ becomes closer to the equilibrium value for all larger t. It is "unstable" when close-by values become farther away. By inspecting the picture, as t gets bigger, it's clear that $f = -1$ is a stable equilibrium, but $f = 2$ is unstable.

3. (4 pts) Set up, but *do not solve*, a differential equation modeling the following rather unlikely situation. Be sure to specify the meanings of your symbols: you should provide a "dictionary" that translates the interesting quantities into your chosen variables ("h = height," for instance). Also be sure to specify the initial condition for the problem.

   > Dr. Gregory House's evil twin Dr. Geoffrey Condo breaks into the set of *Grey's Anatomy*, hoping to murder his brother's arch rival Dr. Meredith Grey. He successfully restrains her to a hospital bed, and implants her with an intravenous drug drip, which he programs to administer an increasing supply of painkillers: 100 miligrams this hour, then 200 mg, and so on, so that the input rate of painkiller is proportional to the amount of time since she was incapacitated. Dr. Condo hopes that by doing this, Dr. Grey will overdose on painkillers.
   >
   > But Dr. Grey's kidneys decide to try to keep her from dying, and filter out the painkiller (assuming the IV also administers plenty of water at the right salinity). In particular, in an hour, her kidneys can filter out one tenth of the total painkillers in her body. How much painkiller is in Dr. Grey's body after a given amount of time?

   We let $P(t)$ be the amount of painkillers in Dr. Grey's body after a given time t. Then to find a differential equation we remember that the net increase of painkillers is equal to the amount in minus the amount out. Painkillers come in at a rate that's increasing by 100 mg/hr every hour, so rate in is $(100\,\mathrm{mg/hr^2})\,t$, and every hour Dr. Grey expels 10% of the total P, so rate out is $(0.1/\mathrm{hr})\,P$. Thus the overall differential equation:

   $$\frac{dP}{dt} = (100\,\mathrm{mg/hr^2})\,t - (0.1/\mathrm{hr})\,P$$

   We also specify the initial conditions: at time $t = 0$, Dr. Condo has just begun his devious plan, so $P(0) = 0$.