Math 1B Quiz #11

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Name: ______________________________________

1. (3 pts) Solve the following initial-value problem, to write f as a function of t:

$$\frac{df}{dt} = t \cdot \left(f^2 + 1\right), \quad f(2) = 0$$

This equation is separable, so we multiply and divide, and then plug in our initial value:

$$\begin{aligned}
\frac{df}{dt} &= t \cdot \left(f^2 + 1\right) \\
\frac{df}{f^2 + 1} &= t\, dt \\
\arctan f &= \frac{t^2}{2} + C \\
f &= \tan\left(\frac{t^2}{2} + C\right) \\
\arctan 0 &= \frac{2^2}{2} + C \\
0 &= 2 + C \\
C &= -2 \\
f(t) &= \tan\left(\frac{t^2}{2} - 2\right)
\end{aligned}$$

2. (3 pts) Solve the following differential equation, to find a one-parameter family of solutions for y as a function of x:

$$2xy' + y = 6x$$

We use the method of finding a multiplier, in order to recognize the equation as a separable one:

$$\begin{aligned}
y' + \frac{1}{2x}y &= 3 \\
\int \frac{dx}{2x} &= \frac{1}{2}\ln x \\
e^{\frac{1}{2}\ln x} &= \sqrt{x} \\
\sqrt{x}\left(y' + \frac{1}{2x}y\right) = \left(\sqrt{x}\,y\right)' &= 3\sqrt{x} \\
\sqrt{x}\,y &= 2x^{3/2} + C \\
y &= 2x + C/\sqrt{x}
\end{aligned}$$

3. (4 pts) Carbon has two stable isotopes — carbon-12 (^{12}C) and carbon-13 (^{13}C) — and one relatively common radioactive isotope carbon-14 (^{14}C), produced in the upper atmosphere by bombardment with cosmic radiation. Plants absorb atmospheric carbon, and hence the concentration of ^{14}C in plants is equal to the atmospheric concentration. When plants die, they do not absorb any new carbon. The amount of ^{14}C in archeological samples is used to date archeological sites.

   (a) Like all radioactive materials, ^{14}C decays at a constant relative rate: the amount that decays in any given period of time is proportional to the amount present. Write a differential equation modeling the amount of ^{14}C in a given amount of time.

   Let $f(t)$ be the amount of ^{14}C after time t. Then

$$\frac{df}{dt} = -kf$$

   We write $-k$ for the coefficient, since we know that the amount of ^{14}C is decreasing, and this lets us use a positive k. Remark: the solution to this differential equation is $f(t) = f(0)e^{-kt}$.

   (b) The half-life of ^{14}C is 5730 years, and the atmospheric concentration of ^{14}C is 600 billion atoms per mole (roughly one part per trillion). What is the solution to your differential equation (relating how much time has elapsed with the amount of ^{14}C left)? You do not need to simplify, but you do need to use units.

   We have $f(t) = f(0)e^{-kt}$. Before any ^{14}C has decayed, i.e. at time $t = 0$, it is at $f(0) =$ 600 billion atoms per mole. In 5730 years, the total ^{14}C has halved: $1/2 = e^{-k\,5730\text{years}}$, so $k = \ln(2)/5730$. Thus

$$f(t) = \left(600 \times 10^9\, rmatoms/mole\right) e^{-(\ln(2)/5730\,\text{yrs})\,t}$$

   (c) A sample from Fell's Cave, in southern Chile, has a ^{14}C concentration of 150 billion atoms per mole. Roughly what is the date of the archeological site?

   By solving the equation above for $f(t) = 150$ billion atoms per mole, we get the right t. Faster: for the ^{14}C to decay from 600 billion atoms per mole to 150 billion atoms per mole requires two halvings: thus, two half lifes $= 11460$ years have passed, so the sample is from roughly 9000 B.C.E.

(d) What is the rate of radioactive decay of ${}^{14}\mathrm{C} \to {}^{12}\mathrm{C}$ in the sample if the current concentration of ${}^{14}\mathrm{C}$ is 150 billion atoms per mole? You do not need to simplify, but you do need to report units.

We recall that $f'(t) = -kf(t)$. Then if $f(t) = 150$ billion atoms per mole, we must have $f'(t) = -(\ln(2)/5730\,\mathrm{yrs}) \times \big(150 \times 10^9\,\mathrm{atoms/mole}\big)$.