Math 1B Quiz #12

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Name: ____

1. (3 pts) Given that $y = \arctan(x)$ is a solution to the following differential equation, and find the most general solution:

$$y'' - y' - 12y = \frac{(1+x)^2}{(1+x^2)^2} - 12\arctan(x)$$

We know that the general solution to any linear differential equation is

$$y_g = y_c + y_p$$

where y_p *is some (any: you choose) particular solution to the differential equation, and* y_c *is the general solution to the corresponding homogeneous equation. So, since* $\arctan x$ *is a solution, we can pick* $y_p = \arctan x$*, and then we have only to solve for* y_c*:*

$$\begin{aligned} 0 &= y_c'' - y_c' - 12y_c \\ 0 &= r^2 - r - 12 \\ &= (r-4)(r+3) \\ r &= 4 \text{ or } -3 \\ y_c &= c_1 e^{4x} + c_2 e^{-3x} \\ y_g &= c_1 e^{4x} + c_2 e^{-3x} + \arctan(x) \end{aligned}$$

2. (3 pts) Solve the initial value problem:

$$y'' + 2y' + 5y = 0, \;\; y(0) = 1, \;\; y'(0) = -3$$

The differential equation is linear homogeneous with constant coefficients. We begin by solving for the general solution. Then we fix the parameters based on the initial values.

$$\begin{aligned}
y'' + 2y' + 5y &= 0 \\
r^2 + 2r + 5 &= 0 \\
r &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 5}}{2} \\
&= -1 \pm \sqrt{1-5} \\
&= -1 \pm \sqrt{-4} \\
&= -1 \pm 2\sqrt{-1} \\
y &= e^{-x}\left(c_1 \cos(2x) + c_2 \sin(2x)\right)
\end{aligned}$$

$$\begin{aligned}
1 = y(0) &= e^{-0}\left(c_1 \cos(2 \cdot 0) + c_2 \sin(2 \cdot 0)\right) \\
&= c_1 \\
-3 = y'(0) &= \Big[-e^{-x}\left(c_1 \cos(2x) + c_2 \sin(2x)\right) \\
&\quad + e^{-x}\left(-2c_1 \sin(2x) + 2c_2 \cos(2x)\right)\Big]_{x=0} \\
&= -c_1 + 2c_2 \\
&= -1 + 2c_2 \\
-2 &= 2c_2 \\
y &= e^{-x}\left(\cos(2x) - \sin(2x)\right)
\end{aligned}$$

3. (4 pts) Solve the differential equation:

$$y'' - y' - 2y = 10\sin(x) + 4x$$

We know that the general solution is $y = y_c + y_p$. *We begin by solving the homogeneous equation:*

$$\begin{aligned} r^2 - r - 2 &= (r-2)(r+1) \\ y_c &= c_1 e^{2x} + c_2 e^{-x} \end{aligned}$$

If we are clever, we find a particular solution by guessing:

$$\begin{aligned} y_p &= A\cos(x) + B\sin(x) + Cx + D \\ y_p'' - y_p' - 2y_p &= (-A\cos(x) - B\sin(x)) - (-A\sin(x) + B\cos(x) + C) \\ &\quad - 2\,(A\cos(x) + B\sin(x) + Cx + D) \\ &= (-A - B - 2A)\cos(x) + (-B + A - 2B)\sin(x) + (-2C)x + (-C - 2D) \\ -3A - B &= 0 \\ A - 3B &= 10 \\ -2C &= 4 \\ -C - 2D &= 0 \\ B &= -3A \\ 10 &= A - 3(-3A) = 10A \\ A &= 1 \\ B &= -3 \cdot 1 = -3 \\ C &= -2 \\ 2 - 2D &= 0 \\ D &= 1 \\ y_p &= \cos(x) - 3\sin(x) - 2x + 1 \end{aligned}$$

Thus, putting everything together, we get

$$y = y_c + y_p = \boxed{c_1 e^{2x} + c_2 e^{-x} + \cos(x) - 3\sin(x) - 2x + 1}$$

Alternately, we could calculate this integral by varying our parameters. Since $y_1 = e^{2x}$ *and* $y_2 = e^{-x}$ *are annihilated by the left hand side of the differential equation, they are a convenient basis in which to work this problem. So we declare that the general solution is*

$$y = u_1 e^{2x} + u_2 e^{-x}.$$

Experience with these problems suggests that we should also demand of our solution that

$$u_1' e^{2x} + u_2' e^{-x} = 0.$$

Then plugging y into the differential equation and simplifying gives

$$u_1'(2e^{2x}) + u_2'(-e^{-x}) = \sin(x) + 4x$$

Solving these two equations for u_1' and u_2' and integrating by parts yields

$$\begin{aligned}
u_1' &= \frac{(10\sin(x) + 4x)e^{-x}}{2e^{2x}e^{-x} - (-1)e^{-x}e^{2x}} \\
&= \frac{(10\sin(x) + 4x)e^{-x}}{3e^{x}} \\
u_2' &= \frac{(10\sin(x) + 4x)e^{2x}}{-3e^{x}} \\
u_1 &= \frac{1}{3}\int (10\sin(x) + 4x)e^{-2x}\,dx \\
&= \frac{10}{3}\int \sin(x)e^{-2x}\,dx + \frac{4}{3}\int xe^{-2x}\,dx
\end{aligned}$$

$$\begin{aligned}
\int \sin(x)e^{kx}\,dx &= -\cos(x)e^{kx} + \int \cos(x)ke^{kx}\,dx \\
&= -\cos(x)e^{kx} + \sin(x)ke^{kx} - \int \sin(x)k^2e^{kx}\,dx \\
(k^2+1)\int \sin(x)e^{kx}\,dx &= -\cos(x)e^{kx} + \sin(x)ke^{kx} \\
\int \sin(x)e^{kx}\,dx &= \frac{k\sin(x) - \cos(x)}{k^2+1}\,e^{kx} + C
\end{aligned}$$

$$\begin{aligned}
\int xe^{kx} &= \frac{x}{k}e^{kx} - \int \frac{1}{k}e^{kx}\,dx \\
&= \frac{kx-1}{k^2}\,e^{kx} + C
\end{aligned}$$

$$\begin{aligned}
u_1 &= \left(\frac{10}{3}\frac{-2\sin(x) - \cos(x)}{5} + \frac{4}{3}\frac{-2x-1}{4}\right)e^{-2x} + c_1 \\
u_2 &= \frac{-10}{3}\int \sin(x)e^{x}\,dx + \frac{-4}{3}\int xe^{x}\,dx \\
&= \left(\frac{-10}{3}\frac{\sin(x) - \cos(x)}{2} + \frac{-4}{3}\frac{x-1}{1}\right)e^{x} + c_2 \\
y &= u_1e^{2x} + u_2e^{-x}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{10}{3}\frac{-2\sin(x)-\cos(x)}{5}+\frac{4}{3}\frac{-2x-1}{4}\right)+c_1e^{2x} \\
&\quad+\left(\frac{-10}{3}\frac{\sin(x)-\cos(x)}{2}+\frac{-4}{3}\frac{x-1}{1}\right)+c_2e^{-x} \\
&= \boxed{-3\sin(x)+\cos(x)-2x+1+c_1e^{2x}+c_2e^{-x}}
\end{aligned}
$$

Sure enough, both methods give the same answer.