Math 1B Quiz #13

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Name: ________________________________

1. (7 pts total) Let's consider a spring with mass = 1 grams and spring-strength = 25 grams per second per second. Then the displacement $y(t)$ of the spring from its equilibrium position satisfies the differential equation

$$(1\text{ g})\frac{d^2y}{dt^2} + (25\text{ g/s}^2)y = F_{\text{ext}}(t)$$

where $F_{\text{ext}}(t)$ is the external force applied to the spring at time t. *(Note: throughout this problem, I will work in cgs units. You may ignore units, if you choose.)*

(a) (1 pt) If the external force is $F_{\text{ext}} = 0$, what are two linearly independent possible movements of the spring?

The characteristic equation is $r^2 + 25 = 0$. *Thus* $\boxed{\cos(5t)}$ *and* $\boxed{\sin(5t)}$ *are both solutions.*

(b) (2 pt) If the external force is $F_{\text{ext}} = \sin(\omega t)$ dynes, and $\omega \neq 5$ Hz, and the spring starts at rest ($y(0) = 0$ and $y'(0) = 0$), what will its position be after time t?

We guess a particular solution of the form $y_p = A\sin(\omega t) + B\cos(\omega t)$. *Then*

$$-A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t) + 25A\sin(\omega t) + 25B\cos(\omega t) = \sin(\omega t)$$

and so $B = 0$ *and* $A = 1/(25-\omega^2)$. *Putting together with* y_c *allows us to solve for our initial conditions:*

$$\begin{aligned}
y &= c_1\sin(5t) + c_2\cos(5t) + \frac{1}{25-\omega^2}\sin(\omega t)\\
0 &= c_1\cdot 0 + c_2\cdot 1 + \frac{1}{25-\omega^2}\cdot 0 = c_2\\
y' &= 5c_1\cos(5t) - 5c2\sin(5t) + \frac{\omega}{25-\omega^2}\cos(\omega t)\\
0 &= 5c_1 + \frac{\omega}{25-\omega^2}\\
c_1 &= -\frac{\omega/5}{25-\omega^2}\\
y &= \boxed{\frac{1}{125-5\omega^2}\left(-\omega\sin(5t) + \sin(\omega t)\right)}
\end{aligned}$$

(c) (2 pt) What if $\omega = 5$ Hz? Then what is the general solution to the differential equation? What is the behavior of the spring as $t \to \infty$?

When $\omega = 5$, the guess in part (b) fails (see, for example, the division by $25-\omega^2$). Thus we guess

$$y_p = At\sin(5t) + Bt\cos(5t)$$

and solve:

$$\begin{aligned}
y_p' &= A\sin(5t) + B\cos(5t) + 5At\cos(5t) - 5Bt\sin(5t) \\
y_p'' &= A\cos(5t) - B\sin(5t) + 5A\cos(5t) - 5B\sin(5t) - 25At\sin(5t) - 25Bt\cos(5 \\
\sin(5t) = y_p'' + 25y &= 6A\cos(5t) - 6B\sin(5t) - 25At\sin(5t) - 25Bt\cos(5t) + 25At\sin(5t) + 25B \\
&= 6A\cos(5t) - 6B\sin(5t) \\
A &= 0 \\
B &= -1/6 \\
y_p &= -\frac{1}{6}t\cos(5t) \\
y = y_c + y_p &= \boxed{c_1\sin(5t) + c_2\cos(5t) - \frac{1}{6}t\cos(5t)}
\end{aligned}$$

As $t \to \infty$, the bounded y_c parts will become very small in comparison to $-\frac{1}{6}t\cos(5t)$, then the amplitude of the spring's oscillation will grow linearly with t.

(d) (1 pt) If we introduce a damping force with coefficient = 6 grams per second, then the differential equation becomes

$$(1\text{ g})\frac{d^2y}{dt^2} + (6\text{ g/s})\frac{dy}{dt} + (25\text{ g/s}^2)y = F_{\text{ext}}(t)$$

What are two linearly independent solutions in the case when $F_{\text{ext}} = 0$?

The characteristic equation becomes $r^2+6r+25$, which solutions $r = -3\pm4\sqrt{-1}$. Thus $\boxed{e^{-3t}\sin(4t)}$ and $\boxed{e^{-3t}\cos(4t)}$ are both solutions.

(e) (1 pt) In this damped case, can any external force of the form $F_{\text{ext}} = \sin(\omega t)$ lead to the kind of resonance as in part (c)? Why or why not?

In the damped case, there can be no resonance: the resonance comes from the extra factor of t required in the guess, which is required because $F_{\text{ext}}(t) = \sin(5t)$ is a solution to the homogenous equation. Since no solution to the homogenous damped equation is of the form $\sin(\omega t)$, no such external force will ever need an extra factor of t in the guessed particular solution.

2. (8 pts total) Consider the linear first-order differential equation

$$(x-1)y' + 2y = g(x)$$

(a) (1 pt) When $g(x) = 0$ the equation is separable. Solve this equation for $y(x)$.
We solve:

$$\begin{aligned}
(x-1)y' + 2y &= 0 \\
(x-1)\frac{dy}{dx} &= -2y \\
\frac{dy}{y} &= \frac{-2\,dx}{x-1} \\
\ln y &= -2\ln(x-1) + c \\
y &= \frac{C}{(x-1)^2}
\end{aligned}$$

(b) (2 pt) Let $g(x) = x + 3$ and solve the differential equation.
This is first-order linear, so we use the method of multiplying by an integrating factor:

$$\begin{aligned}
(x-1)y' + 2y &= x+3 \\
y' + \frac{2}{x-1}y &= \frac{x+3}{x-1} \\
e^{\int \frac{2\,dx}{x-1}} &= e^{2\ln(x-1)} \\
&= (x-1)^2 \\
\left((x-1)^2 y\right)' = (x-1)^2 y' + (x-1)^2\frac{2}{x-1} &= (x-1)^2\frac{x+3}{x-1} = (x-1)(x+3) \\
(x-1)^2 y &= \int (x-1)(x+3)\,dx \\
&= \frac{x^3}{3} + x^2 - 3x + C \\
y &= \frac{\frac{x^3}{3} + x^2 - 3x}{(x-1)^2} + \frac{C}{(x-1)^2}
\end{aligned}$$

(c) (2 pt) Now let's consider power-series solutions. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Write the left-hand-side of the equation in terms of series, and manipulate the expression so that it is of the form $\sum_{n=0}^{\infty} A_n x^n$. (So A_n will be some expression involving n, c_n, c_{n+1}, etc.)
We recall that $y'(x) = \sum_{n=0}^{\infty} nc_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$. *Then the left-hand side is*

$$xy' - y' + 2y = x\sum_{n=0}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty}\left((n+2)c_n - (n+1)c_{n+1}\right)x^n$$

(d) (1 pt) If $g(x) = x + 3$, write a recursion relation defining c_{n+1} in terms of c_n. *Hint: The first two formulas, defining c_1 in terms of c_0 and defining c_2 in terms of c_1, are different from the rest of the formulas.*

We compare coefficients order-by-order between the expression from (c) and $3 + x + 0x^2 + 0x^3 + \ldots$. The constant terms give $\boxed{2c_0 - c_1 = 3}$. *At order x we have* $\boxed{3c_1 - 2c_2 = 1}$. *And, for $n \geq 2$, we have* $\boxed{(n+2)c_n - (n+1)c_{n+1} = 0}$.

(e) (1 pt) Find an explicit formula for c_n, depending only on n and c_0.

We solve the equations from part (d):

$$\begin{aligned}
c_1 &= 2c_0 - 3 \\
c_2 &= \frac{3}{2}c_1 - \frac{1}{2} \\
&= 3c_0 - 5 \\
c_n &= \frac{n+1}{n}c_{n-1} \\
&= \frac{n+1}{n}\frac{n}{n-1}c_{n-2} = \frac{n+1}{n-1}c_{n-2} \\
&= \frac{n+1}{n-2}c_{n-3} \\
&= \ldots \\
&= \frac{n+1}{3}c_2 \\
&= (n+1)\left(c_0 - \frac{5}{3}\right)
\end{aligned}$$

(f) (1 pt) Use the formula from part (d), or use any other method, to calculate the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$.

We use the ratio test: if $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$, then $\sum a_n$ converges, and if the limit is larger than 1, then the series diverges. Thus, to study $\sum c_n x^n$, we compute

$$\lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| |x|$$

But by (c), we have

$$\frac{c_{n+1}}{c_n} = \frac{n+2}{n+1} \underset{n\to\infty}{\longrightarrow} 1$$

and so the series converges if $|x| < 1$ and diverges if $|x| > 1$; i.e. the radius of convergence is 1.