Math 1B Section 112 Quiz #7

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Name: ______________________________________

1. **True or False** (1 pt each) For each of the following statements, decide if it is true or false. You do not need to show work: I will grade only your answers.

   (a) Let's say $0 \le a_n \le b_n$, and $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. If $A = B$, then $a_n = b_n$ for every n.
   TRUE. The first inequality guarantees that $\sum (b_n - a_n) = B - A = 0$ is a sum of nonnegative terms, but the only way to add up nonnegative numbers and get 0 is if each of the numbers is itself 0.

   (b) If $0 \le a_n \le f(n)$, where $f(x)$ is a continuous decreasing function on $x \in [1, \infty)$, such that $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.
   FALSE. Although this looks like a cross between the integral and comparison tests, all we know is that the series is less than a divergent integral, so we don't know whether the series converges or diverges.

   (c) If $0 \le a_n \le \pi/2$ and $\sum_{n=1}^{\infty} \tan(a_n)$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
   TRUE. This follows from the comparison test, since $\tan(x) > x$ if $\pi/2 > x > 0$, and so $a_n < \tan(a_n)$.

For the next two questions, use either the **Limit Comparison Test** or the **Integral Test** to determine if the series converges or diverges. Be sure to check that the series satisfies the conditions necessary for the test.

2. (3 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n)}{n+\ln(n)}$

   We recall that $\lim_{n\to\infty} \arctan(n) = \pi/2$. *Then comparing with* $1/n$ *(which diverges) gives*

   $$\lim_{n\to\infty} \frac{\arctan(n)/(n+\ln n)}{1/n^2} = \lim_{n\to\infty} \arctan(n)\frac{1}{1+(\ln(n)/n)} = \frac{\pi}{2}\cdot 1$$

   which is more than 0 *and less than* ∞. *So since* $\sum 1/n$ *converges, so must our series* $\sum \arctan(n)/(n+\ln n)$.

3. (4 pts) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\,\ln n\,(\ln(\ln n))^2}$

   Recognizing this as something integrable, we check the conditions for the integral test:

   - $1/(x\,\ln x\,(\ln(\ln x))^2)$ *is always positive.*
   - *It is the product of three decreasing functions* ($1/x$, $1/\ln x$, *and* $1/(\ln\ln x)^2$), *so must itself be decreasing.*

   Thus, using the integral test, we check

   $$\int_3^{\infty} \frac{dx}{x\,\ln x\,(\ln(\ln x))^2} = \int_{\ln\ln 3}^{\infty} \frac{du}{u^2} = \frac{1}{\ln\ln 3} < \infty$$

   where we substitute $u = \ln(\ln x)$. *Thus the original series also converges.*