Math 1B Section 112 Quiz #9

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Name: ____

1. (3 pts) Write
$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2+n} = \frac{1}{2} + \frac{x}{6} + \frac{x^2}{12} + \frac{x^3}{20} + \frac{x^4}{30} + \frac{x^5}{42} + \dots$$
in terms of elementary functions. *(Hint: Partial fractions)* For what values of x is your solution justified?

This problem was a worksheet problem that I ran out of space for. I was interested in seeing how folks would solve it. Ultimately I decided to make the problem out of 2 points.

We can start by finding the interval of convergence (if the series doesn't converge, then any manipulations we do won't be justified). By the ratio test, this series converges if
$$\lim_{n\to\infty} \left| \frac{x^n}{(n+1)^2+(n+1)} \frac{n^2+n}{x^{n-1}} \right| = |x| < 1$$
Moreover, by comparison with $\sum 1/n^2$, *this series converges absolutely at the endpoints* ± 1. *(If you got this far, I gave 2 points; 1 for radius of convergence.)*

How do you write the series in terms of nice functions?
$$\begin{aligned}
\frac{1}{n^2+n} &= \frac{1}{n} - \frac{1}{n+1} \text{ by partial fractions} \\
\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2+n} &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} - \sum_{n=1}^{\infty} \frac{x^{n-1}}{n+1} \\
&= \frac{1}{x} \int_0^x \sum_{n=1}^{\infty} t^n \, dt - \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} \\
&= \frac{1}{x} \int_0^x \frac{dt}{1-t} - \frac{1}{x^2} \sum_{2}^{\infty} \frac{x^n}{n} \\
&= -\frac{1}{x} \ln(1-x) - \frac{1}{x^2} \left(-x + \sum_{1}^{\infty} \frac{x^n}{n} \right) \\
&= -\frac{1}{x} \ln(1-x) + \frac{1}{x} + \frac{1}{x^2} \ln(1-x)
\end{aligned}$$

This argument is justified whenever everything is defined and all the sub-series converge. I.e. not at $x = 1$ or $x = 0$. But we can interpret $-\frac{1}{x}\ln(1-x)+\frac{1}{x}+\frac{1}{x^2}\ln(1-x)$ as being $1/2$ when $x = 0$ (this is the limit, in any case), and the $x = 1$ end-point must be correct by continuity.

2. (3 pts) Find the Taylor series expansion of $\cos(x)$ centered at $c = \pi/2$. What is the interval of convergence for this series?

There is a sneaky way to do this problem:

$$\cos(x) = -\sin\left(x - \pi/2\right) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(x - \pi/2\right)^{2n+1}, \textit{ which converges everywhere.}$$

But here's the non-sneaky way:

n	$\cos^{(n)}(x)$	$\sin^{(n)}(\frac{\pi}{2})$
0	$\cos(x)$	0
1	$-\sin(x)$	-1
2	$-\cos(x)$	0
3	$\sin(x)$	1
4	$\cos(x)$	0
5	$-\sin(x)$	-1
6	$-\cos(x)$	0
7	$\sin(x)$	1
$\vdots$	$\vdots$	$\vdots$
$2n$		0
$2n+1$		$(-1)^{n+1}$
$\vdots$	$\vdots$	$\vdots$

Thus, the Taylor series for $\sin(x)$ *centered at* $\pi/2$ *is*

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x - \pi/2\right)^{2n+1}$$

and the ratio test gives that this converges everywhere.

3. (4 pts) For the following power series

(a) find the general nth term (i.e. write it as $\sum_{n=0}^{\infty}$(something))

(b) find the radius of convergence

(c) check whether the series converges at the endpoints

so that you can determine for which x the series

- converges absolutely
- converges conditionally
- diverges.

$$\frac{1}{4} + \frac{2x}{9} + \frac{3x^2}{16} + \frac{4x^3}{25} + \frac{5x^4}{36} + \frac{6x^5}{49} + \dots$$

We recognize the pattern as $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{(n+2)^2} x^n$. *Then the ratio test gives*

$$\lim_{n\to\infty} \left| \frac{(n+2)x^{n+1}}{(n+3)^2} \times \frac{(n+2)^2}{(n+1)x^n} \right| = \lim_{n\to\infty} \frac{(n+2)^3}{(n+1)(n+3)^2} |x| = 1 \cdot |x|$$

So this series $\boxed{\textit{converges absolutely when } |x| < 1}$ *and* $\boxed{\textit{diverges when } |x| > 1}$.

Checking endpoints, we see that when $x = -1$ the series converges by the alternating series test. At $\boxed{x = 1 \textit{ the series diverges}}$ *(for instance, by limit comparison test with* $\sum \frac{1}{n}$*). Thus,* $\boxed{\textit{the convergence at } x = -1 \textit{ must be conditional}}$.