Math 1B Quiz #13

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Name: ______________________________________

1. (7 pts total) Let's consider a spring with mass = 1 grams and spring-strength = 25 grams per second per second. Then the displacement $y(t)$ of the spring from its equilibrium position satisfies the differential equation

$$(1\text{ g})\frac{d^2y}{dt^2} + (25\text{ g/s}^2)y = F_{\text{ext}}(t)$$

where $F_{\text{ext}}(t)$ is the external force applied to the spring at time t. *(Note: throughout this problem, I will work in cgs units. You may ignore units, if you choose.)*

(a) (1 pt) If the external force is $F_{\text{ext}} = 0$, what are two linearly independent possible movements of the spring?

(b) (2 pt) If the external force is $F_{\text{ext}} = \sin(\omega t)$ dynes, and $\omega \neq 5$ Hz, and the spring starts at rest ($y(0) = 0$ and $y'(0) = 0$), what will its position be after time t?

(c) (2 pt) What if $\omega = 5$ Hz? Then what is the general solution to the differential equation? What is the behavior of the spring as $t \to \infty$?

(d) (1 pt) If we introduce a damping force with coefficient $= 6$ grams per second, then the differential equation becomes

$$(1\text{ g})\frac{d^2y}{dt^2} + (6\text{ g/s})\frac{dy}{dt} + (25\text{ g/s}^2)y = F_{\text{ext}}(t)$$

What are two linearly independent solutions in the case when $F_{\text{ext}} = 0$?

(e) (1 pt) In this damped case, can any external force of the form $F_{\text{ext}} = \sin(\omega t)$ lead to the kind of resonance as in part (c)? Why or why not?

2. (8 pts total) Consider the linear first-order differential equation

$$(x-1)y' + 2y = g(x)$$

(a) (1 pt) When $g(x) = 0$ the equation is separable. Solve this equation for $y(x)$.

(b) (2 pt) Let $g(x) = x + 3$ and solve the differential equation.

(c) (2 pt) Now let's consider power-series solutions. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Write the left-hand-side of the equation in terms of series, and manipulate the expression so that it is of the form $\sum_{n=0}^{\infty} A_n x^n$. (So A_n will be some expression involving n, c_n, c_{n+1}, etc.)

(d) (1 pt) If $g(x) = x + 3$, write a recursion relation defining c_{n+1} in terms of c_n. *Hint: The first two formulas, defining c_1 in terms of c_0 and defining c_2 in terms of c_1, are different from the rest of the formulas.*

(e) (1 pt) Find an explicit formula for c_n, depending only on n and c_0.

(f) (1 pt) Use the formula from part (d), or use any other method, to calculate the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$.