Math 1B Worksheet 10: Limits of sequences

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Theo Johnson-Freyd
http://math.berkeley.edu/~theojf/

Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

1. (a) Let's say a sequence s_n diverges to $+\infty$. What is the limit of $1/s_n$ as $n \to \infty$? Justify your answer.

   (b) Is there a sequence t_n such that $\lim_{n\to\infty} t_n = 0$, such that $t_n \neq 0$ for any n, and $1/t_n$ does not diverge to $+\infty$ nor to $-\infty$?

   (c) If you know that $t_n > 0$ for every n and that $t_n \to 0$ as $n \to \infty$, then what can you say about $\lim_{n\to\infty} 1/t_n$?

2. (a) Does the sequence

$$\{a_n\}_{n=1}^{\infty} = \{0, 1, 0, 1, 0, 1, 0, 1, \dots\} = \left\{\frac{1}{2} + \frac{1}{2}(-1)^n\right\}_{n=1}^{\infty}$$

   have a limit? Why or why not? If it does, what is it?

   (b) Let's take the sequence b_n of averages:

$$\begin{aligned} b_1 &= a_1 &&= 0 \\ b_2 &= \frac{a_1 + a_2}{2} &&= \frac{1}{2} \\ b_3 &= \frac{a_1 + a_2 + a_3}{3} &&= \frac{1}{3} \\ b_4 &= \frac{a_1 + a_2 + a_3 + a_4}{4} &&= \frac{1}{2} \end{aligned}$$

   Does this sequence have a limit? If so, what is it?

(c) What if you take averages again? What is the limit of the sequence c_n?

$$\begin{aligned} c_1 &= b_1 \\ c_2 &= \frac{b_1+b_2}{2} \\ c_3 &= \frac{b_1+b_2+b_3}{3} \\ c_4 &= \frac{b_1+b_2+b_3+b_4}{4} \end{aligned}$$

Of course, if $a_n = \{0,1,0,1,\ldots\}$ is going to have any "limit", then certainly $\tilde{a}_n = \{1,0,1,0,\ldots\}$ should have the same limit. How does this justify the limit-average you found?

3. Repeat the previous exercise with the sequence

$$A_n = \{1,-1,2,-2,3,-3,4,-4,\ldots\}$$

Does this sequence converge? Does it converge after taking averages once? Twice? What is the limit (after taking averages)?

In fact, we can think of a_n as a sequence of partial sums: $a_1 = 1-1$, $a_2 = 1-1+1$, $a_3 = 1-1+1-1$, $a_4 = 1-1+1-1+1$, etc. Then A_n is the sequence of partial sums of the previous sequence "squared":

$$\begin{aligned} (1)^2 &= \underline{1} \\ (1-1)^2 &= \underline{1-2}+1 \\ &\quad \text{(by the usual } (a+b)^2 = a^2+2ab+b^2\text{)} \\ (1-1+1)^2 &= \underline{1-2+3}-2+1 \\ (1-1+1-1)^2 &= \underline{1-2+3-4}+3-2+1 \\ &\quad \text{etc.} \end{aligned}$$

Then $A_1 = 1$, $A_2 = 1-2$, $A_3 = 1-2+3$, $A_4 = 1-2+3-4$, and so on. How does this justify the limit-averge of A_n?

4. The following sequence diverges, and it still diverges after you take averages once (why?). But it does converge after taking averages twice.

$$\{\alpha_n\} = \{0,1,\underbrace{0,0}_{2},\underbrace{1,1}_{2},\underbrace{0,0,0,0}_{4},\underbrace{1,1,1,1}_{4},\ldots,\underbrace{0,\ldots,0}_{2^j},\underbrace{1,\ldots,1}_{2^j},\underbrace{0,\ldots,0}_{2^{j+1}},\ldots\}$$

Can you turn this into an example of a sequence that diverges no matter how many times you take averages?