# Math 1B Worksheet 11:
# Infinite Series

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

1. Here are two facts:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \ln 2$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \frac{\pi}{4}$$

   Use these facts to sum the following series:

   (a) $2 - \frac{1}{2} - \frac{1}{4} + \frac{2}{5} - \frac{1}{6} - \frac{1}{8} + \frac{2}{9} - \frac{1}{10} - \frac{1}{12} + \frac{2}{13} - \dots$

   (b) $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \dots$

   (c) $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \frac{1}{9\cdot 10} - \frac{1}{11\cdot 12} + \dots$

   (d) $\frac{1}{1\cdot 2} + \frac{1}{5\cdot 6} + \frac{1}{9\cdot 10} + \frac{1}{13\cdot 14} + \frac{1}{17\cdot 18} + \dots$

2. Remember how to evaluate the geometric series: if $|r| < 1$, then we can evaluate $S = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$ by mutiplying by $r$ and subtracting:

$$\begin{array}{rrcl} & S & = & a + ar + ar^2 + ar^3 + \dots \\ - \ ( & r\cdot S & = & \quad ar + ar^2 + ar^3 + \dots \ ) \\ \hline & S - rS & = & a \\ \\ \text{thus} & S & = & \frac{a}{1-r}. \end{array}$$

Use this method to compute the following sums:

(a) $1 + \frac{2}{3} + \frac{3}{9} + \frac{4}{27} + \frac{5}{81} + \frac{6}{243} + \dots$

(b) $1 + \frac{4}{2} + \frac{9}{4} + \frac{16}{8} + \frac{25}{32} + \frac{36}{64} + \dots$

3. This exercise provides a proof that there are infinitely many prime numbers. A *prime number*, like 2, 3, 5, or 101, is a positive number with exactly two positive factors. This exercise uses the fact that any positive number can be written as the product of prime numbers in exactly one way.

   (a) "Expand out" the following product into a single series:

   $$\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots\right)$$

   What is the rule that determines whether a fraction does or does not appear in the (expanded out) sum? What is the value of the total sum?

   (b) How about the triple product:

   $$\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots\right)\left(1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \dots\right)?$$

   (c) What if you were to do this for *every* prime number:

   $$\left(1 + \frac{1}{2} + \dots\right)\left(1 + \frac{1}{3} + \dots\right)\left(1 + \frac{1}{5} + \dots\right)\left(1 + \frac{1}{7} + \dots\right)\left(1 + \frac{1}{11} + \dots\right)\dots$$

   (d) Use the integral test to check if the total "expanded out" sum converges or diverges.

   (e) Each of the individual multiplicands has a finite value (what is it?). Thus, conclude that there must be infinitely many multiplicands, and hence infinitely many prime numbers.

Incidentally, here's another fact I could have listed in exercise 1:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \dots = \frac{\pi^2}{6}$$

One can follow the steps above, replacing each $\left(1 + \frac{1}{p} + \left(\frac{1}{p}\right)^2 + \left(\frac{1}{p}\right)^3 + \dots\right)$ with $\left(1 + \frac{1}{p^2} + \left(\frac{1}{p^2}\right)^2 + \left(\frac{1}{p^2}\right)^3 + \dots\right)$. Then at the end, the fact that $\pi^2/6$ is irrational gives another proof that there are infinitely many primes.