Math 1B Worksheet 11:
Infinite Series

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

1. **Geometric series**

   (a) What is the sum
   $$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots ?$$
   How about
   $$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots ?$$
   If k is some number strictly greater than 1, what is
   $$\sum_{n=1}^{\infty}\left(\frac{1}{k}\right)^n = \frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\ldots ?$$

   (b) Now, let's use the integral test. Let k be an integer greater than 2. Draw a picture: is $\displaystyle\int_{x=0}^{\infty}\frac{dx}{k^x}$ an overestimate or underestimate of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{k^n}$? Based on the picture, estimate by how much it is an under- or overestimate

   (c) Integrate
   $$\int_{x=1}^{\infty}\frac{dx}{k^x}$$

   One way of interpreting these answers is that "in sequence-land, $\ln x = x-1$."

2. **Telescoping series**

   (a) Does every telescoping series converge? Use partial fractions decomposition to "sum" the following series:

   $$\sum_{n=1}^{\infty} \frac{n^2+3n+1}{n^2+3n+2}$$

   (b) Is this summation justified? Use the divergence test to show that this series actually diverges.

   (c) We can write any series as a telescoping sum. Write out the first few terms of

   $$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots = (b_1-b_2)+(b_2-b_3)+(b_3-b_4)+\ldots$$

   I.e., pick a b_1 and b_2 so that $b_1 - b_2 = 1$. For instance, maybe $b_1 = 2$ and $b_2 = 1$. Then figure out what b_3 must be so that $b_2 - b_3 = 1/2$. Then find b_4 so that the next term is $1/3$. And so on. What happens to b_n as n gets large?

   (d) Say for some series $\sum_{n=1}^{\infty}$, we have $A_n = B_n - B_{n+1}$, such that $A_n \geq 0$ and $b_n \geq 0$ for every n. Use this to write an inequality relating B_n and B_{n+1}.

   (e) Thus, check the assumptions of the Monotonic Sequence Theorem, and use it to conclude that the sequence b_n has a limit L.

   (f) Write out the partial sum

   $$S_n = A_1 + A_2 + \ldots + A_n$$

   in terms of the B_n. What is

   $$\lim_{n\to\infty} S_n \text{ ?}$$

This shows that a sum of *positive* numbers, which is realizable as a telescoping sum so that every term is a difference of two *positive* numbers, definitely converges.