Math 1B Worksheet 13: Comparison Tests, Alternating Series, etc.

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

1. Do the following series converge? Why or why not?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1 + (-1)^n n + (1/2)^n n^2}{n^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

2. For what values of p do the following series converge?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{p^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^p}\right)$

(e) $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{(-1)^n}{n^p}\right)$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^p}{p^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{p^n}{n^p}$

3. (a) Find a sequence $\{a_n\}$ so that $\sum_{n=1}^{\infty} a_n$ diverges, but $\sum_{n=1}^{\infty} (a_n)^2$ converges.
   (b) Find a sequence $\{a_n\}$ so that $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} (a_n)^2$ diverges.

4. Often we use integrals to test whether series converge or not. Sometimes it's useful to go the other way: use the Alternating Series theorem to prove that $\displaystyle\int_0^{\infty} \frac{\sin(x)}{x}\, dx$ converges. **Warning:** a priori, this series could diverge both at $x \to \infty$ and at $x = 0$ (since we can't divide by 0, and so $\frac{\sin(x)}{x}$ isn't well-defined at the end-point); why does the integral actually pose no problems at $x = 0$?