Math 1B Worksheet 13:
Ratio Test

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GSI: Theo Johnson-Freyd
http://math.berkeley.edu/~theojf/

Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

1. Remember that $e = \lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge or diverge?

2. For what values of p does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$
   (a) converge absolutely?
   (b) converge conditionally?
   (c) diverge?

3. For what values of p and r do the following series converge absolutely?
   (a) $\displaystyle\sum_{n=1}^{\infty} r^n\, n^p$
   (b) $\displaystyle\sum_{n=1}^{\infty} r^n\, (n!)^p$

4. (a) Show that, for any polynomial $f(x) = f_d x^d + f_{d-1} x^{d-1} + \ldots + f_1 x + f_0$, the ratio test fails to show that $\sum_{n=1}^{\infty} f(n)$ diverges. What's a test that does work?
   (b) Show that, for any polynomial $f(x)$, the series $\sum_{n=1}^{\infty} r^n f(n)$ converges if $|r| < 1$.

5. For any number q (for "quantum"), we can define the "nth q-integer" $[n]_q$ by

$$[n]_q = q^{n-1} + q^{n-2} + \ldots + q + 1.$$

So, for instance, $[n]_1 = n$. Then the q-factorial is defined by

$$[n]_q! = [n]_q \cdot [n-1]_q \cdot \ldots \cdot [2]_q \cdot [1]_q$$

(a) If $q > 0$, show that $\sum_{n=1}^{\infty} \frac{1}{[n]_q!}$ converges.

(b) If $q = 1/2$, does $\sum_{n=1}^{\infty} \frac{3^n}{[n]_q!}$ converge or diverge? What if $q = 1$?

6. What if you didn't know the ratio test? In this exercise, you'll discover that you could have invented the ratio test yourself.

   (a) Let's say you have a series $\sum a_n$ and you want to know if it converges or not. First you might decide that you only care about absolute convergence, so let's assume that $a_n \geq 0$ for every n. You know that the geometric series converges: for what r does $\sum br^n$ converge?

   (b) Let's try to use the limit comparison test: if $\lim_{n\to\infty} a_n/(br^n) = L$, where L is finite and bigger than 0, then what do you know about whether $\sum a_n$ converges or diverges?

   (c) Let's try to guess what r makes this limit finite and not zero. Write down the same limit with n replaced by $n+1$. Now combine these limits to get rid of the b, and then solve for r.

   (d) Your answer to part (c) depended on whether r was more or less than 1. Use this to guess the Ratio Test.

   (e) Let's now check the guess. Say $\lim_{n\to\infty} a_{n+1}/a_n = l < 1$. Define $r = (1+l)/2$. Check that $r < 1$. Conclude that, for large enough n (say for $n \geq N$ for some N), we definitely have $a_{n+1} < ra_n$. (Why?)

   (f) Thus, write down an inequality using just a_n, a_N, and r, that holds for $n \geq N$.

   (g) Use this and the Comparison Test to prove that $\sum a_n$ converges.

   (h) Repeat (and re-write) steps (e) through (g) if $l > 1$.

   (i) This method doesn't always work. Show that there is no r so that the limit $\lim_{n\to\infty}(1/n)/(br^n) = L$ where L is finite and not 0.