Math 1B Worksheet 15: Power Series

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GSI: Theo Johnson-Freyd
http://math.berkeley.edu/~theojf/

Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

1. Let's start with some practice: For what x do the following series (i) converge absolutely? (ii) converge conditionally? (iii) diverge? *(Hint: everything inside the interval of convergence is absolute; everything outside diverges. The boundaries you don't know: for each boundary point, decide if it's absolute, conditional, or divergent.)*

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{3^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-a)^n}{b^n}$ where a and b are fixed (but unknown) real numbers, and $b > 0$.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$ *(Hint: remember that* $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.*)*

2. (a) What is the interval of convergence for

$$f(x) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \ldots?$$

   (b) What is the value of $f(x)$? There are two ways to do this problem (so pick one):

      i. Multiply $f(x)$ by x, and subtract $f(x) - xf(x)$. Do you recognize this power series? Evaluate $f(x) - xf(x)$ and use that to solve for $f(x)$.

      ii. Integrate $F(x) = \int_0^x f(t)\,dt$ term-by-term. Do you recognize this power series? Evaluate $F(x)$ and differentiate to get $f(x) = F'(x)$.

3. I've reproduced the last five problems from today's homework. Pick one to work on as a group. Time permitting, at the end of the class some groups will present their solutions:

   (a) **[11.8.36]** If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+4} = c_n$ for all $n \geq 0$, find the interval of convergence of the series and a formula for $f(x)$.

   (b) **[11.8.37]** Show that if $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$, where $c \neq 0$, then the radius of convergence of the power series $\sum c_n x^n$ is $R = 1/c$.

   (c) **[11.8.38]** Suppose that the power series $\sum c_n(x-a)^n$ satisfies $c_n \neq 0$ for all n. Show that if $\lim_{n\to\infty} |c_n/c_{n+1}|$ exists, then it is equal to the radius of convergence f the power series.

   (d) **[11.8.39]** Suppose the series $\sum c_n x^n$ has radius of convergence 2 and the series $\sum d_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\sum (c_n + d_n)x^n$?

   (e) **[11.8.40]** Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?