Math 1B Worksheet 16: Manipulating Power Series

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

$$\boxed{\text{Reminder: } 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}, \text{ if } |x| < 1.}$$

1. (a) Write down the power series expansion of $f(x) = 1/(1 + x)$.
   (b) Integrate both sides of your equation from part (a) to get a power series expansion of $\ln(1 + x)$. What is the radius of convergence? Does this function converge at the endpoints? Use this to write down a series for $\ln(2)$.
   (c) What is the integral of $\ln(1 + x)$ in terms of functions (not power series)? Now, use the power series from (b) to get a power series for this integral. How does this compare to the power series you get if you integrate the series in (b)?
2. Integrate the power series for $1/(1 + x^2)$. What's the radius of convergence of your series? Does it converge at the endpoints? What equation do you get when you substitute $x = 1$? What about when $x = 1/\sqrt{3}$?
3. (a) What power series do you get if you differentiate the (boxed, above) series for $1/(1 - x)$?
   (b) What power series do you get if you differentiate again?
   (c) How would you write $\sum_{n=0}^{\infty} n^2 x^n$ as a function?
   (d) If $p(x)$ is any polynomial, use the ratio test to determine the radius of convergence of $\sum_{n=0}^{\infty} p(n)x^n$. Does this converge on the boundary?
   (e) Come up with a method that you could use to write $\sum_{n=0}^{\infty} p(n)x^n$ as a function for any given polynomial $p(n)$. For example, what is $\sum_{n=0}^{\infty}(3n^2 - 4n + 1)x^n$?

4. This last problem is about analytic continuation, in which you continue a function outside its radius of convergence. The full theory of analytic continuation relies on the theory of complex numbers.

   (a) What is the interval of convergence of

   $$1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 + \ldots?$$

   What does the expression equal?

   (b) Thus, what is the interval of convergence of

   $$1 + \frac{x-1}{2} + \left(\frac{x-1}{2}\right)^2 + \left(\frac{x-1}{2}\right)^3 + \left(\frac{x-1}{2}\right)^4 + \ldots?$$

   (c) Expand out these expressions and combine like terms. You will have an infinite summation as the coefficient c_n for each power x^n. Guess the rule, and write $c_n = \sum_{j=0}^{\infty}$(something).

   (d) Does the series for c_n converge? Evaluate the sum to get explicit formulæ for c_n.

   (e) What is the interval of convergence of $\sum_{n=0}^{\infty} c_n x^n$? Do you recognize this sum? *(Hint: assuming the arithmetic is correct, you should get a geometric series. Which one?)*

   (f) By evaluating the series in (b) to get a function, check your work to see how it compares to the function in part (e).