# Math 1B Worksheet 17: Taylor Series

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

1. Write out the Taylor series expansion[1] of the following functions. What is the radius of convergence?
   (a) $f(x) = 1/(2-x)$
   (b) $f(x) = \sin(2x - \pi/4)$
2. What is the Taylor series expansion of $e^{-1/x^2}$ centered at 0? What is the radius of convergence of this series? Why should this make you troubled?
3. By using the Taylor series expansion for $\sin(x)$ only up to the cubic term, find the non-zero solution for
$$x^2 = \sin(x)$$
4. Prove that
$$\int_0^1 \frac{1+x^{30}}{1+x^{60}} dx = 1 + \frac{c}{31}$$
where $0 < c < 1$
5. Soon (after the midterm) we will start talking about solving differential equations (although secretly we've solved some already). Here's a way to approximately solve a differential equation with Taylor series: Let's say you want to find a function $f(x)$ such that
$$f'(x) = 2f(x) - 1 \text{ and } f(0) = 1$$

[1] When mathematicians say "Taylor Series Expansion" without specifying where to center it, they usually mean to expand around 0, and I will adopt that convention here.

What is $f'(0)$? What about $f''(0)$? Write out the first five terms of the Taylor series, and then guess the pattern for the Taylor series, and check your guess. What is the function $f(x)$?

6. Write the Taylor series of $e^x$. Use this to *define* $e^{t\frac{d}{dx}}$. Let $f(x)$ is an analytic function (*analytic* means "equal to its Taylor series", so $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\, x^n/n!$) with infinite radius of convergence.

$$g(t, x) = e^{t\frac{d}{dx}} f(x)$$

and, using the Taylor series for $f(x)$, simplify $g(t, x)$. Interpret this geometrically.