Math 1B Worksheet 19: Introducing Differential Equations

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Don't forget to draw pictures.

1. Show that
$$y(x) = 2\sec(x) + \tan(x)$$
satisfies the differential equation
$$\frac{dy}{dx} = 2\sec(x)\,y(x) - 3\sec^2(x)$$

2. Show that $y(x) = C\sec(x) + C\tan(x)$ is a one-parametric family of solutions to the differential equation $y' = y\sec(x)$. Given the boundary condition $y(\pi/4) = 5$, what is C?

3. Considered the differential equation $y'(x) = x/y$. If $y(1) = 1$, what can you say about the slope of y? For what values of x and y is the slope 0? In what regions of the x, y-plane is y' negative? What are the possible linear solutions $y = mx + b$?

4. Consider the Haber-Bosch process for creating ammonia from hydrogen and nitrogen:
$$3H_2 + N_2 \to 2NH_3$$
   (a) Assuming that the concentration of NH_3 starts at zero (and that no other reactions occur), write down two (algebraic) equations: one relating the concentration of $NH_3(t)$ to the concentration of $H_2(t)$, and one relating it to the concentration of $N_2(t)$. Each of your equations will really be a one-parametric family of equations. What are the parameters? Let's assume that the starting concentrations of H_2 is three times the starting concentration of N_2, so that there are exactly the right amounts of both chemicals: $H_2(0) = 3N_2(0) = C$.

(b) Since three H_2 molecules must bump into one N_2 molecule for the reaction to occur, the rate of increase of NH_3 is proportional to the concentration of N_2 and to the cube of the concentration of H_2. Write a differential equation describing the concentration of NH_3. This equation is also really a one-parametric family of equations: there's a proportionality constant K that acts as an unknown parameter.

(c) Show that

$$\mathrm{NH}_3(t) = A - Bt^{1/3}$$

is a two-parametric family of solutions to your differential equation from part (b). How do the parameters A and B relate to the original parameters C and K in parts (a) and (b)?

(d) In fact, the Haber-Bosch process is not very favorable, and the model in this problem is only accurate if the concentration of NH_3 is very small (compared to the concentrations of N_2 and H_2). Because the reaction can also go the other way: $2\mathrm{NH}_3 \to 3\mathrm{H}_2 + \mathrm{N}_2$. So really the equations for the rate of increase of NH_3 is the equation you wrote down in (b), minus a term proportional to the square of $\mathrm{NH}_3(t)$. Write the correct differential equation describing the production of ammonia.

(e) Let's say you don't know anything about the relative concentrations of the three chemicals. Based on your equation in part (d), write down the equilibrium condition: what relationship must the concentrations of the chemicals satisfy for there to be no net production of ammonia? (How many parameters are in your solution? In fact, this parameter depends on the temperature.) For what concentrations of there net ammonia production, and for what concentrations does more ammonia turn into hydrogen and nitrogen?