Math 1B Worksheet 20:
Separable Differential Equations

Thursday, 6 November 2007

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Don't forget to draw pictures.

1. Here are some differential equations to warm up with:

   (a) $y' = (x^n y) \,/\, ((\ln y)^n)$

   (b) $y' = \cos^2(y/x) + y/x$

2. Consider the differential equation

$$\frac{dy}{dx} = 2\sqrt{y}$$

   What is the general solution to this equation? What is the equation for y if $y(3) = 1$? By direct substitution, check that your equation really is a solution.

   What is the equation for y if $y(3) = 0$? Draw the direction field corresponding to this differential equation.

3. Last time, we discussed the production of ammonia

$$\mathrm{N_2} + 3\mathrm{H_2} \to 2\mathrm{NH_3}$$

   When the concentration of ammonia is small ($\mathrm{NH_3}(0) = 0$), the rate of reaction is controlled by the concentrations of nitrogen and hydrogen:

$$\frac{d\mathrm{NH_3}(t)}{dt} = K\left(\mathrm{N_2}(0) - \frac{1}{2}\mathrm{NH_3}(t)\right)\left(\mathrm{H_2}(0) - \frac{3}{2}\mathrm{NH_3}(t)\right)^3$$

(a) If $H_2(0) = 3N_2(0)$ (so that there is no limiting reactant), solve the differential equation explicitly. In terms of K, how long does it take to convert half of the reactants to ammonia?

(b) If $H_2(0) > 3N_2(0)$ (so that ammonia is limiting), what's the behavior of the system? What if hydrogen is limiting?

4. Myoglobin, like hemoglobin, allows oxygen to dissolve in water. (The normal solubility is about 0.0001 mol O_2 per liter.) Hemoglobin transports oxygen through your blood; myoglobin stores oxygen in muscle tissue.

$$\mathrm{Mb}(aq) + \mathrm{O}_2(g) \leftrightarrow \mathrm{Mb(O_2)}(aq)$$

Write and solve a differential equation to describe the concentration of oxygenated myoglobin, assuming that this is the only reaction that occurs. Remember that the reaction can occur in both directions.

5. The acceleration $a = dv/dt$ on an object is equal to the total force F on the object divided by the object's mass m. A raindrop falling through the air experiences a constant downward force $F_{\text{gravity}} = mg$ from gravity, and an upward force from air resistance proportional to the velocity $F_{\text{resistance}} = kv$. Terminal velocity occurs when the net acceleration is zero.

(a) In terms of m, g, and k, solve for the terminal velocity.

(b) Write and solve a differential equation describing the velocity of a raindrop that starts with zero velocity at time $t = 0$.

(c) What if a particle starts with a downward velocity greater than terminal velocity, e.g. because it was thrown down?

A marble falling through honey also has a terminal velocity, but in a viscous material, the resistance is proportional to the square of the velocity $F_{\text{resistance}} = lv^2$. (The effective gravity is also reduced by buoyancy, but it is still a constant.) Repeat parts (a), (b), and (c) above in this setting.

6. And, as always, the requisite turkey problem:

A turkey, which starts at 70 degrees Fahrenheit, is placed in a 300-degree. The rate of heat transfer between two materials is proportional to the difference in temperatures of the materials (the proportionality constant depends on various parameters, like the type of material and the size). The turkey is done when the internal temperature reaches 180 degrees. If after an hour the internal temperature of the turkey is 86 degrees, and dinner is at 5 p.m., when should you start cooking for Thanksgiving?