Math 1B Worksheet 22: Linear Differential Equations

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Don't forget to draw pictures.

1. Warm-up: Solve the differential equation

$$x^2 y' + 2xy = \cos^2 x$$

2. In the quiz last time, we suggested that a particular evil villain might try to murder someone by injecting them with painkillers. The painkillers were administered at an increasing rate: after time t (in hours), the input was $t \times 100\,\mathrm{mg}/\mathrm{hr}^2$. The victim's body expelled the drugs proportional to the amount in her system: if $P(t)$ is the amount of drug in her body at time t, then she expelled the drug at $0.1\,P/\mathrm{hr}$. Thus, the overall differential equation is

$$\frac{dP}{dt} = (100\,\mathrm{mg}/\mathrm{hr}^2)\,t - (0.1/\mathrm{hr})\,P$$

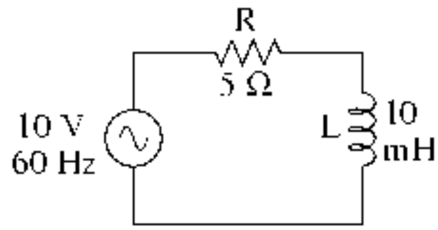
with initial value $P(0) = 0$. Solve this linear differential equation. If death sets in when the painkiller concentration reaches $15\,000\,\mathrm{mg}$, how long does the rescue team have?

3. Remember that a series circuit with resistance R, inductance L, and power source $E(t)$ satisfies the linear differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

where $I(t)$ is the current around the circuit. An AC power source provides a sinosoidal voltage: e.g. for this circuit, the power supply provides a voltage $E(t) = (10\,\mathrm{V})\sin(t\times$

60 Hz). Solve the differential equation to determine the current flowing around the loop at a given time.



4. Cynthia (Theo's officemate) gets homework assigned every day, in roughly a sinosoidal pattern: in particular, the amount of new homework (measured in hours) she has to do is roughly

$$\left(\frac{1}{2}\,\mathrm{hr}\right)(1 - \cos(2\pi t\,/\mathrm{week}))$$

where t is the amount of time since the beginning of the semester (a Sunday). If every day Cynthia finishes half the homework she has left, how much of her homework does she still need to do at the end of the semester (after 15 weeks)?

Theo, a slacker, only takes classes that don't collect homework.

5. In the textbook, the suggestion for how to solve a linear first-order differential equation is to multiply by an integrating factor. Let's consider the differential equation

$$y' + xy = x^3$$

What is the general solution to this differential equation?

Pick two solutions: $y_1(x)$ and $y_2(x)$, say. What is $y_1(x) - y_2(x)$? Let's call this function $y_0(x)$. Does $y_0(x)$ satisfy our original differential equation? What differential equation does $y_0(x)$ satisfy?

Guess the general story: if we solve the differential equation $y' + r(x)\,y = q(x)$, and pick two solutions y_1 and y_2, what differential equation will the difference $y_1 - y_2 = y_0$ satisfy? Try to prove your guess.

When we start solving second-order linear differential equations, we won't have integrating factors to multiply by. Instead, we will do the backwards order of this problem: we will solve the (simpler) differential equation satisfied by y_0, and then find some particular y_1. Then any other solution y_2 will be $y_1 + C\,y_0$. (Except these will be second-order, so there will really be a two-parameter family of solutions y_0 to the simpler equation: $y_0 = A\,a(x) + B\,b(x)$, so $y = y_1(x) + A\,a(x) + B\,b(x)$ will be a two-parameter family of solutions.)