Math 1B Worksheet 23:
Linear homogeneous second-order differential equations

Thursday, 15 November 2007

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/∼theojf/

Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Don't forget to draw pictures.

1. **Physics: dampened springs**

   (a) A frictionless spring is described by a second-order differential equation: the force (mass times acceleration) is proportional to the displacement of the spring. If the mass is m and the "spring constant" (constant of proportionality) is k, write and solve a differential equation to find the most general equation for the position of the spring. (You need to know whether k is positive or negative: draw a picture to figure out which direction the force should push.)

   (b) Often engineers place springs in viscous fluids in order to dampen the movement of the spring; if the fluid is jostled enough so that the internal flow is consistently turbulent, then the damping force will be proportional to the velocity (let's say with proportionality constant c). If $c^2 > 4mk$, what is the behavior of the spring?

   (c) If $c^2 < 4mk$, then the solution to the differential equation is

   $$\text{position} = e^{-ct/2m}\left(A\cos(t\omega) + B\sin(t\omega)\right)$$

   By plugging into your differential equation, find the frequency ω.

   Sketch the graph of this solution. Car shock absorbers are dampened springs; if your shock absorber has $c^2 > 4mk$, what would it feel like to go over a bump?

2. **Mathematics: linear independence**

   A differential operator — for example, $\frac{d^2}{dx^2} + 2\frac{d}{dx} + 6$ — is like a function. It is a function, in fact, but it takes *functions* to *functions*, whereas a normal function takes *numbers* to *numbers*. Consider a function on three variables:

$$F(x, y, z) = x + 2y + 6z$$

Find two linearly independent solutions to $F(x, y, z) = 0$.

What is "linear independence"? Two triples (x_1, y_1, z_1) and (x_2, y_2, z_2) are *linearly dependent* if there's some number α such that $x_1 = \alpha x_2$, $y_1 = \alpha y_2$, and $z_1 = \alpha z_2$, and *linearly independent* if there is no such number.

$F(x, y, z)$ is *linear*, meaning that

$$F(x_1 + x_2, y_1 + y_2, z_1 + z_2) = F(x_1, y_1, z_1) + F(x_2, y_2, z_2)$$

(Prove this!) Find a two-parameter family of solutions to $F(x, y, z) = 0$.

3. **Mathematics: when the characteristic equation has one real roots**

   We can analyze a linear homogeneous differential equation with constant coefficients by looking at its characteristic equation:

   $$ay'' + by' + c = 0 \quad \rightsquigarrow \quad ar^2 + br + c = 0$$

   If this equation has two real roots r_1 and r_2, then the general solution is $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.

   Let's fix $C_1 = C_2$. Find

   $$\lim_{r_1 \to r_2} y(x)$$

   Now, let's set $r_1 - r_2 = \epsilon$, and $C_1 = 1/\epsilon$ and $C_2 = -1/\epsilon$. Use L'Hopital's rule to calculate

   $$\lim_{\epsilon \to 0} y(x)$$

4. **Mathematics: hyperbolic functions**

   If the characteristic equation has zero real roots, then the solution in general is of the form

   $$e^{\alpha x} \left(A \cos(\beta x) + B \sin(\beta x)\right)$$

   If there are two real roots, we can use the same equation, but with the hyperbolic functions rather than the trigonometric ones. For what values of α and β is

   $$y(x) = e^{\alpha x} \left(A \cosh(\beta x) + B \sinh(\beta x)\right)$$

   a two-parameter family of solutions to

   $$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

   if $b^2 > 4ac$?