Math 1B Worksheet 24: Some mathematics

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Don't forget to draw pictures.

1. **Linear Independence**

   A differential operator — for example, $\frac{d^2}{dx^2} + 2\frac{d}{dx} + 6$ — is like a function. It is a function, in fact, but it takes *functions* to *functions*, whereas a normal function takes *numbers* to *numbers*. Consider a function on three variables:

   $$F(x, y, z) = x + 2y + 6z$$

   Find two linearly independent solutions to $F(x, y, z) = 0$.

   What is "linear independence"? Two triples (x_1, y_1, z_1) and (x_2, y_2, z_2) are *linearly dependent* if there's some number α such that $x_1 = \alpha x_2$, $y_1 = \alpha y_2$, and $z_1 = \alpha z_2$, and *linearly independent* if there is no such number.

   $F(x, y, z)$ is *linear*, meaning that

   $$F(x_1 + x_2, y_1 + y_2, z_1 + z_2) = F(x_1, y_1, z_1) + F(x_2, y_2, z_2)$$

   (Prove this!) Find a two-parameter family of solutions to $F(x, y, z) = 0$.

2. **When the characteristic equation has one real roots**

   We can analyze a linear homogeneous differential equation with constant coefficients by looking at its characteristic equation:

   $$ay'' + by' + c = 0 \quad \rightsquigarrow \quad ar^2 + br + c = 0$$

If this equation has two real roots r_1 and r_2, then the general solution is $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.

Let's fix $C_1 = C_2$. Find

$$\lim_{r_1 \to r_2} y(x)$$

Now, let's set $r_1 - r_2 = \epsilon$, and $C_1 = 1/\epsilon$ and $C_2 = -1/\epsilon$. Use L'Hopital's rule to calculate

$$\lim_{\epsilon \to 0} y(x)$$

3. **Hyperbolic functions**

If the characteristic equation has zero real roots, then the solution in general is of the form

$$e^{\alpha x} \left(A \cos(\beta x) + B \sin(\beta x) \right)$$

If there are two real roots, we can use the same equation, but with the hyperbolic functions rather than the trigonometric ones. For what values of α and β is

$$y(x) = e^{\alpha x} \left(A \cosh(\beta x) + B \sinh(\beta x) \right)$$

a two-parameter family of solutions to

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

if $b^2 > 4ac$?