Math 1B Worksheet 25:
Second-order linear nonhomogeneous differential equations

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Don't forget to draw pictures.

1. What trial solutions would you use for the differential equation $ay'' + by' + cy = g(x)$, if you use the method of "undetermined coefficients"?
   (a) $g(x) = \sinh(x)$
   (b) $g(x) = \sin(x)\cos(2x)$
   (c) $g(x) = \tan(x)\sin(2x)$
   (d) If $g(x)$ is a solution to $ag''(x) + bg'(x) + cg(x) = 0$?
2. Find the general solution to the differential equation $y'' + 6y' + 9y = x^2 e^{-3x}$.
3. (a) If you know trial solutions for $ay'' + by' + cy = g_1(x)$ and $ay'' + by' + cy = g_2(x)$, what is a trial solution for $ay'' + by' + cy = g_1(x) + g_2(x)$?
   (b) What about for $ay'' + by' + cy = g_1(x)g_2(x)$?
4. Let's say we're trying to solve $ay'' + by' + cy = g(x)$ by the method of variation of parameters, and let's say that $b^2 > 4ac$, so that the characteristic equation has two real solutions $y_i = e^{r_i x}$ ($i = 1$ or 2). We're looking for $y_p(x) = u_1(y)y_1(x) + u_2(x)y_2(x)$. Explicitly solve the system of linear equations you get for $u_1'(x)$ and $u_2'(x)$ (in terms of $g(x)$, a, and r_1 and r_2).
5. Find the general solution to the differential equation:

$$y'' + 4y' + 3y = \frac{1}{1 + e^{2x}}$$

6. Here's another way to solve a second-order linear equation with constant coefficients. I will sketch the method in general; try it with the specific equation $4y'' - 4y' + y = e^x$ (which you could also solve by "undetermined coefficients"):

   (a) We're trying to find solutions $y = y(x)$ to $ay'' + by' + cy = g(x)$. Begin by finding a non-zero solution (you just need one) $y_0(x)$ to the homogeneous equation $ay_0'' + by_0' + cy_0 = 0$.

   (b) Now say we have some solution y to the nonhomogeneous equation. Whatever it is, we can divide by $y_0(x)$ to write $y(x) = z(x)\, y_0(x)$. Plug $y = zy_0$ into the nonhomogeneous equation, and use the product rule to expand out the derivatives.

   (c) Use the fact that y_0 is a solution to the homogeneous equation to simplify the expression, and combine like terms in derivatives of z. Notice that there are no terms in z (just in z' and z'').

   (d) Let $w(x) = z'(x)$; then your equation becomes a *first-order linear* differential equation in w. Solve this equation for $w(x)$.

   (e) Integrate this solution to get $z(x)$, and multiply by $y_0(x)$ to get $y(x)$.