Math 1B Worksheet 27:
A short review of series and sequences

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Don't forget to draw pictures.

1. If you know that $\sum_{n=0}^{\infty} c_n x^n$ converges when $n = 2$ and diverges when $n = 3$, what can you say about the interval of convergence of $\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$? What about $\sum_{n=0}^{\infty} \frac{1}{n} c_{n-1} x^n$ (where we define $c_{-1} = 0$)?

   (Hint: what functions do these series represent, in terms of $f(x) = \sum_{n=0}^{\infty} c_n x^n$? What series, written in the form $\sum_0^{\infty} a_n x^n$, represents the function $x f'(x)$? What function represents the series $\sum_0^{\infty} c_{n-1} x^n$?)

2. Find the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n$$

   This problem appeared on the second midterm in Zworski's 1B class ("the other 1B"). It was too hard: finding the radius of convergence is not impossible, but to check the endpoints requires something akin to *Stirling's Formula*:

$$\lim_{n\to\infty} \frac{n! e^n}{n^n \sqrt{n}} = \sqrt{2\pi}$$

   which can be restated as

$$n! \approx \sqrt{2\pi n} (n/e)^n$$

   You may or may not also need limits like $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$; this one you may be expected to know. In general, $\lim_{n\to\infty} \left(\frac{n+x}{n}\right)^n = e^x$.

3. Prove that

$$\lim_{n\to\infty}\left(\frac{n+x}{n}\right)^n = e^x$$

by way of a differential equation: differentiate the left hand side, and show that it satisfies the differential equation $f'(x) = f(x)$, which e^x also satisfies. Check that it has the same initial value $f(0) = 0$.

4. (a) If you know that $\sum a_n^2$ converges, do you know that $\sum a_n^3$ necessarily converges? Why or why not?

   (b) In fact, it's possible for $\int_0^\infty f(x)^2\,dx$ to converge but for $\int_0^\infty f(x)^3\,dx$ to diverge. Why does this not disprove the similar statement about sums? How are integrals and sums different?

   (c) On the other hand, if $|f(x)| < 1$ for every $x \geq 0$, use the integral comparison test and the absolute convergence test to show that if $\int_0^\infty f(x)^2\,dx$ converges, then so does $\int_0^\infty f(x)^3\,dx$.