Math 1B Worksheet 27: Convergence of integrals, series, and sequences

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Don't forget to draw pictures.

1. (a) Does
$$\int_{1/2}^{2} \frac{dx}{(x^5 - x)^{1/3}}$$
converge or diverge?

   (b) Does
$$\int_{0}^{\infty} \frac{dx}{(x^5 - x)^{1/3}}$$
converge or diverge?

2. (a) Show that
$$\int_{1}^{\infty} \frac{\sin(\pi x)}{x^2}\, dx$$
converges.

   (b) By comparing with the appropriate series and using the alternating series test, show that (a) is in fact still true with the x^2 replaced by any x^p with $p > 0$.

   (c) By comparing with the appropriate series, show that
$$\int_{1}^{\infty} \frac{\sin(\pi \sqrt{x})}{x}\, dx$$
diverges.

3. (a) If you know that $\sum a_n^2$ converges, do you know that $\sum a_n^3$ necessarily converges? Why or why not?

   (b) In fact, it's possible for $\int_0^\infty f(x)^2\,dx$ to converge but for $\int_0^\infty f(x)^3\,dx$ to diverge. Why does this not disprove the similar statement about sums? How are integrals and sums different?

   (c) On the other hand, if $|f(x)| < 1$ for every $x \geq 0$, use the integral comparison test and the absolute convergence test to show that if $\int_0^\infty f(x)^2\,dx$ converges, then so does $\int_0^\infty f(x)^3\,dx$.

4. Solve for x:

$$x^{\left(x^{\left(x^{(\dots)}\right)}\right)} = 2$$

   Is your answer reasonable? What numbers could replace "2" in this problem to make the final answer converge?

5. (a) Let's say that $\{a_n\}$ is a positive strictly-decreasing sequences that converges to 0, and that $\sum b_n$ converges as a series. Show that

$$\sum a_n b_n$$

   converges.

   (b) Conclude that if $\sum b_n$ converges, then so does $\sum (b_n)^3$.

   (c) On the other hand, find an example of a sequence $\{b_n\}$ where $\sum b_n$ converges but $\sum (b_n)^4$ diverges.

6. By analogy with series, define the "Limit Comparison Test" for integrals. I.e. make sense of the idea that if $f(x) \approx g(x)$, then $\int_0^\infty f(x)\,dx$ converges if and only if $\int_0^\infty g(x)\,dx$ converges. In addition to some limit, what else do you have to assume about f and g for your test to be true?

7. Consider the recurrence relation

$$c_{n+1} = \frac{n+1}{k(n+2)} c_n$$

   (a) If c_n satisfies the above relationship for $n \geq 0$, what is the radius of convergence of

$$\sum_{n=0}^{\infty} c_n x^n$$

   (b) Show that the positive end-point of this interval of convergence diverges, whereas the lower endpoint converges.

   (c) Let $y(x) = \sum_{n=0}^\infty c_n x^n$. Interpret the recurrence relation as a differential equation for y. Solve this differential equation.