Math 1B Worksheet 6: Approximate Integration

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Don't forget to draw pictures.

1. When we make measurements, we get decimal numbers — these are, of course, approximations to "true" values. Good algorithms exist to add, subtract, multiply, divide, and take square roots of decimals.

   But that's about all you can compute easily without calculus, and it's often not a lot. For instance, let's say you wanted to figure out what the value of$\ln(2)$ is to four decimal places (error of 0.00001). You would have to (don't actually do this):

   - Calculate the value of e very accurately (say to six or seven places). $e = \lim_{n\to\infty}(1 + 1/n)^n$, so this requires taking a very large value of n and evaluating this carefully.
   - Guess a value a_1 of $\ln(2)$. Accurately compute e^{a_1} by multiplying and taking square roots. Measure how far it is from 2.
   - Refine your guess, and keep repeating until you get to within your allowable error.

   In general, this takes a very long time. But calculus gives a better way to do this. So actually do this: write down a definite integral that, if evaluated accurately, would yield $\ln(2)$.

   Now calculate how many intervals you would have to divide this integral into in order to approximate $\ln(2)$ to within an error of 0.00001 using the midpoint rule.

2. Every polynomial can be exactly integrated. Do this! Evaluate:

$$\int_0^1 \left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0\right) dx$$

But one way of understanding problem 1 is that not every rational function can be exactly integrated. Similarly, if all we know about are rational functions, we have to invent a new function to evaluate $\int_0^x \frac{dt}{1+x^2}$; this new function is often called $\arctan(x)$.

However, even after we've invented all the trigonometric and exponential functions, and accurately computed their values into a table, so that we can easily compose them, there are still functions that are hard to integrate. For instance, it is a theorem that e^{x^2} does not have an integral that is expressible in elementary functions, and yet it turns up all the time, especially in statistics. So let's invent a new function, called the "Gaussian":

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2} dt$$

(a) In terms of $G(x)$, evaluate

$$\int_a^1 \sqrt{\ln \frac{1}{y}}\, dy$$

where a is some given number with $0 < a < 1$.

(b) Pick your favorite approximation technique. How many numbers would you need to add together to compute $G(1)$ to within a tenth-of-a-percent error (0.001)?

3. Let f be a positive functions with positive first derivative and negative second derivative on $[0, 1]$. For a given n, put in order by size:

$$L_n,\ R_n,\ M_n,\ T_n,\ \textstyle\int_a^b f(x)\, dx.$$

By drawing a picture for $n = 1$, prove that the midpoint rule M_n is a better approximation than the trapezoid rule T_n. Now let f be a quadratic function $f(x) = ax^2 + bx + c$. Prove that M_n is in fact exactly twice as good an approximation as T_n.