Math 1B Worksheet 7: Approximate and Improper Integration

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Don't forget to draw pictures.

1. Say we're interested in approximately integrating

$$\int_a^b f(x)\,dx.$$

Let's divide $[a, b]$ into $2n$ intervals:

$$a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{2n} = b$$

with equal spacing $x_i - x_{i-1} = \Delta x$.

(a) Write down a formula for M_n in terms of f and the x_i. (*Caution:* we have $2n$ intervals, but I only want M_n.)

(b) Write down a formula for T_n in terms of f and the x_i.

We expect M_n to be twice as good an approximation to the integral as is T_n (for instance, compare the error estimates). Moreover, if $f(x)$ is convex (up, say), then we expect M_n to slightly overestimate $\int_a^b f(x)\,dx$, and T_n to slightly underestimate the integral. (If f is convex down, then these are reversed.) So we have

$$M_n \approx \int_a^b f(x)\,dx + E \text{ and } T_n \approx \int_a^b f(x)\,dx - 2E$$

(c) (Approximately) solve for $\int_a^b f(x)\,dx$ in terms of M_n and T_n.

(d) Now use your formulas for M_n and T_n to write $\int_a^b f(x)\,dx$ (approximately) in terms of $f(x_0), \ldots, f(x_{2n})$.

(e) Do you recognize this expression? It very accurately approximates the integral.

2. (a) What is the area of the region

$$x \geq 1,\ y \geq 0, \text{ and } y \leq \frac{1}{x}$$

(b) What is the volume of the figure you get by revolving this region around the x-axis?

In section 8.2 we will see that this infinite cone has infinite surface area. What does this say about how much paint it can hold versus how much you would need to paint the inside wall?

3. How fast must you throw a rock in order for it to escape the gravitational pull of the earth? Remember that an object of mass m traveling at a speed v has kinetic energy

$$KE = \frac{1}{2}mv^2.$$

In order to escape, it must resist the force of gravity, which is

$$F = \frac{GmM}{r^2}$$

where G is a universal constant, M is the mass of the Earth, and r is the distance of the rock from the center of the Earth. Thus, the total gravitational potential energy the rock must overcome is

$$PE = \int_R^\infty F dr$$

where R is the radius of the Earth. The minimum v for which $KE \geq PE$ is called the "escape velocity".

4. The "Laplace Transform" of a function $f(t)$, is the function of s given by

$$\mathcal{L}[f](s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

if this integral converges. Find the Laplace transforms $\mathcal{L}[1](s)$, $\mathcal{L}[t](s)$, and $\mathcal{L}[e^t](s)$. What are the domains of these functions (for what s values do the integrals converge)?

5. Evaluate

$$\int_0^\infty x^n\, e^{-x}\, dx$$

You may want to try this for small values of n first, look for a pattern, and then justify your answer with a reduction formula.

Then use your answer to calculate $\mathcal{L}[x^n](s)$ for any n.