Math 32 Discussion Problems

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In groups, solve and talk about the following problems. Be sure to take turns writing on the chalkboards. Some of these problems area easy: if everyone in your group feels very confident with a type of problem, then there's no reason to work on them, but do take the time to explain the exercises to each other. Some of these problems are hard, so you should not expect to solve all of them.

Plane Geometry: Rectangular Coordinates and Equations of Lines

1. What is the distance between two points (a, b) and (x, y)? Prove that the distance between any two points is never negative, and that it is 0 only when the two points coincide.

2. Remember that we can "add" points on the Cartesian plane: $(a, b) + (x, y) \stackrel{\text{def}}{=} (a + x, b + y)$. Similarly, we can "subtract" points by subtracting their coordinates. Let's define the "absolute value" $\|(x, y)\|$ of a point (x, y) to be the distance from (x, y) to $(0, 0)$. Then prove the *Triangle Inequality* in two dimensions:
$$\|(a, b) + (x, y)\| \leq \|(a, b)\| + \|(x, y)\|$$
Let P, Q, and R be points on the Cartesian plane. If $(a, b) = Q - P$ and $(x, y) = R - Q$, then what is $R - P$? What are the distances between P, Q, and R in terms of (a, b) and (x, y)? What does the above inequality say about the sides of the triangle PQR?

3. Use the distance formula to show that in each case the triangle with the given vertices is isosceles (has two sides of the same length):
(a) $(0, 2)$, $(7, 4)$, $(2, -5)$ (b) $(-1, -8)$, $(0, -1)$, $(-4, 4)$ (c) $(-7, 4)$, $(-3, 10)$, $(1, 3)$

4. The converse of the Pythagorean Theorem says that if a, b, and c are the lengths of the sides of a triangle and $a^2 + b^2 = c^2$, then the triangle has a right angle opposite the side of length c. Use this to determine whether the folllowing triangles are right:
(a) $(4, 5)$, $(-3, 9)$, $(1, 3)$ (b) $(7, -1)$, $(-3, 5)$, $(-12, -10)$ (c) $(-8, -2)$, $(1, -1)$, $(10, 19)$

5. Calculate the slopes of each side of the triangles in the previous problem. Remember that two lines, with slopes m and n, are perpendicular if and only if $mn = -1$. Use this to check your answers to the previous problem.

6. Let's prove the "slopes of perpendicular lines" fact. If $P = (a, b)$ and $Q = (x, y)$ are points, find the slope m of the line connecting them. Now rotate the Cartesian plane by 90°, to get $P' = (-b, a)$ and $Q' = (-y, x)$. What is the slope m' of the line connecting P' and Q'? Check that $mm' = -1$.

7. Use the "slopes of perpendicular lines" fact to prove the Pythagorean Theorem and its converse: Let P, Q, and R be three arbitrary points. Let m be the slope of the line $\overleftrightarrow{PQ}$ and n the slope of the line $\overleftrightarrow{QR}$. Show that $mn = -1$ if and only if $d(P, Q)^2 + d(Q, R)^2 = d(P, R)^2$. (I.e., calculate everything in terms of the coordinates of P, Q, and R.)

Solving Equations

1. Determine whether the given value is a solution to the equation:

   (a) $\frac{1}{x} = \frac{3}{x} - 1$; $x = 2$

   (b) $y + 3 + \frac{2}{y-1} = \frac{2y}{y-1}$; $y = 1$

   (c) $x^2 - 2x - 4 = 0$; $x = 1 + \sqrt{5}$ or $1 - \sqrt{5}$

2. Solve each equation:

   (a) $2m - 1 + 3m + 5 = 6m - 8$

   (b) $\frac{5}{x+2} - \frac{2x-1}{5} = 0$

   (c) $x^3 - 6x^2 + x = 0$

   (d) $y + 3 + \frac{2}{y-1} = \frac{2y}{y-1}$

   (e) $(x+2)(x+1) = x^2 + 11$

   (f) $\frac{x^2 - 3x}{x+1} = \frac{4}{x+1}$

3. Find all real solutions to the following quadratic equations:

   (a) $x^2 + 12x + 18 = 0$

   (b) $3y^2 - 3y - 4 = 0$

   (c) $16s^2 + 8s + 1 = 0$

4. What is the sum of the two roots of $x^2 - 10x + 15 = 0$? What is the product?

5. What are the two roots of $x^2 + (\sqrt{2} - 1)x - \sqrt{2} = 0$? Do not use the quadratic formula.

6. Prove the AM-GM inequality: Let r and s be positive real numbers, and define the "arithmetic mean" to be $m = (r+s)/2$ and the "geometric mean" to be $g = \sqrt{rs}$. In terms of m and g, find a quadratic equation with roots r and s. Now use the fact that this equation has two real roots to prove that $m \geq g$. When is there equality $m = g$?

7. For what value of k does $kx^2 + kx + 1$ have only one real root?

8. Solve the following equations:

   (a) $3x^3 - 48x = 0$

   (b) $t - t^3 = 0$

   (c) $y^4 - 81 = 0$

   (d) $2t^5 + 5t^4 - 12t^3 = 0$

   (e) $x^5 = 36x$

   (f) $x^4 - 5x^2 = -6$

   (g) $(x^2 - 1)^4 - 81 = 0$

   (h) $(x^2 - 1)^4 + 81 = 0$

   (i) $2x^5 - 15x^3 - 27x = 0$

   (j) $y^{-2} - y^{-1} = 0$

   (k) $t^{3/2} = 8$

   (l) $(t+3)^4 = 625$

   (m) $\sqrt{x^2 + 5x - 2} = 2$

   (n) $x - \sqrt{x} = 20$

   (o) $\sqrt{2y-3} - \sqrt{3y+3} + \sqrt{3y-2} = 0$

   (p) $\sqrt{2y-3} + \sqrt{3y+3} + \sqrt{3y-2} = 0$

   (q) $x^{4/3} + 3x^{2/3} - 28 = 0$