Math 32 Discussion Problems

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In groups, solve and talk about the following problems. Be sure to take turns writing on the chalkboards. Some of these problems area easy: if everyone in your group feels very confident with a type of problem, then there's no reason to work on them, but do take the time to explain the exercises to each other. Some of these problems are hard, so you should not expect to solve all of them.

Solving Equations and Inequalities

1. Here's how the Babylonians solved quadratic equations. The Babylonians hadn't invented negative numbers, so they would only find the positive solution. I'll write their method in modern notation; they also didn't have variables or symbols like "+", "=" (indeed, even the idea of "decimal number" was after their time).

$$\begin{aligned}
\text{We want to solve: } x^2 + 8x &= 84 \\
\text{Factor: } x(x+8) &= 84 \\
\text{Define: } t &= \text{Average}(x, x+8) \\
&= x + 4 \\
\text{Substitute: } x(x+8) &= (t-4)(t+4) \\
&= t^2 - 16 \\
\text{Solve: } t^2 - 16 &= 84 \\
t^2 &= 100 \\
t &= 10 \\
\text{Substitute back: } x + 4 &= 10 \\
x &= 6
\end{aligned}$$

Use these steps to solve $x^2 + 14x = 72$ and $x^2 + 2Ax = B$, where $B > 0$.

2. Let's try to solve $\sqrt{4 - \sqrt{4 + x}} - x = 0$.

   (a) Try solving this directly, isolating the radical to get a polynomial. You'll end up with a 4th-order polynomial, which is generally very hard to solve.

   (b) So instead, let's try to solve $\sqrt{k - \sqrt{k + x}} - x = 0$ for an unknown k. Isolate the radical, and write this as a polynomial in terms of k and x.

   (c) Your polynomial should be quadratic in k. Solve for k in terms of x.

   (d) You have two possibilites for k in terms of x, each of which is a quadratic expression in x. Solve each one for x in terms of k, and substitute $k = 4$ back in. You'll get four possible answers for x.

   (e) Substitute your possible answers into the original equation $\sqrt{4 - \sqrt{4 + x}} - x = 0$ to figure out which ones are real solutions and which are extraneous.

3. Solve the following equations for x. Hint: one way to deal with absolute values is to square both sides, since $|y| = \sqrt{y^2}$. Be sure to check for extraneous solutions.

   (a) $|x+6| + 1/2 = 0$ (b) $4|x-2| = 3x-4$ (c) $|x+1| - |3x-2| = 0$

4. Solve the inequality and write the answer using interval notation.

   (a) $6 - 4x \leq 22$ (c) $\frac{1}{4}(x-1) - \frac{1}{5}(2x+3) \leq x$ (c) $\dfrac{2}{3} \leq \dfrac{5-3t}{-2} \leq \dfrac{3}{4}$

   (b) $1 - 2(t+3) - t \leq 1 - 2t$ (d) $-3 \leq 2x+1 \leq 5$

5. Solve the compound inequalities, writing the answer using interval notation:

   (a) $x - 3 < 3x + 1 < 17 - x$ (b) $|3x+5| > 17$ (c) $|x-4| \leq x$

6. (a) Graph the equation $y = |x-1| + |x-2|$.

   (b) Solve the inequality $3 > |x-1| + |x-2|$. You can either use the graph, or you can consider the cases $x < 1$, $1 \leq x < 2$, and $x \geq 2$.

7. (a) Let a and b be nonnegative numbers. Using the fact that a square is always nonnegative, and that squaring preserves order, show that

$$\mathrm{GM} = \sqrt{ab} \leq \frac{a+b}{2} = \mathrm{AM}$$

   (b) Show that if $\sqrt{ab} = (a+b)/2$, then $a = b$.

   (c) Using the fact that $4 = 2 \cdot 2$, and so $\sqrt[4]{x} = \sqrt{\sqrt{x}}$, show that for any four nonnegative real numbers a, b, c, d, we have

$$\sqrt[4]{abcd} \leq \frac{a+b+c+d}{4}$$

   (d) Say you want to build a rectangular enclosure with 60 yards of fence. If the two sides are x and y, what is $\mathrm{AM} = (x+y)/2$? Hence, what is the maximal value for $\mathrm{GM} = \sqrt{xy}$? What is the maximal value for the area? By using 7(b), find the values of x and y that maximize the area.

   (e) Say one side of your enclosure is along a perfectly straight river, and so does not need a fence. Let's let x be the length of the side parallel to the river, and y the length perpendicular to the river. What is $\mathrm{AM} = (x+2y)/2$? Write the area in terms of $\mathrm{GM} = \sqrt{2xy}$. Since the geometric mean only equals the arithmetic mean when a and b are equal, what values of x and y maximize the area?

8. Let $f(x)$ is a continuous function (any function you can write down with $+$, $\times$, $-$, $\div$, $\sqrt{\cdot}$, etc., is continuous wherever it's defined). Then as x changes, the only way $f(x)$ can change from positive to negative is if it is momentarily 0 or undefined. Thus, to solve the inequality $f(x) > 0$, it's enough to find all the numbers x where $f(x) = 0$ or $f(x)$ is undefined, and then test each interval. Solve the following inequalities:

   (a) $x^2 - 3x - 4 \leq 0$ (c) $x^2 + 4x - 32 < 0$ (c) $\dfrac{x+1}{x+2} > \dfrac{x-3}{x+4}$ Hint: combine the fractions.

   (b) $-\frac{1}{2}x^2 = \frac{7}{2}x - 5 < 0$ (d) $2x^3 - 9x^2 + 4x \geq 0$