Math 32 Discussion Problems

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In groups, solve and talk about the following problems. Be sure to take turns writing on the chalkboards. Some of these problems area easy: if everyone in your group feels very confident with a type of problem, then there's no reason to work on them, but do take the time to explain the exercises to each other. Some of these problems are hard, so you should not expect to solve all of them.

Functions and their graphs

1. Give an example of a function from {the people in this room} $\to \mathbb{R}$. What can you say about the range of your function?

2. Given a person x, let $f(x)$ be x's sister's age. Is this a function? Why or why not?

3. What is the domain of the function that takes integers and doubles them? What is the range? Does this function have an inverse? Describe the inverse, including its domain and range.

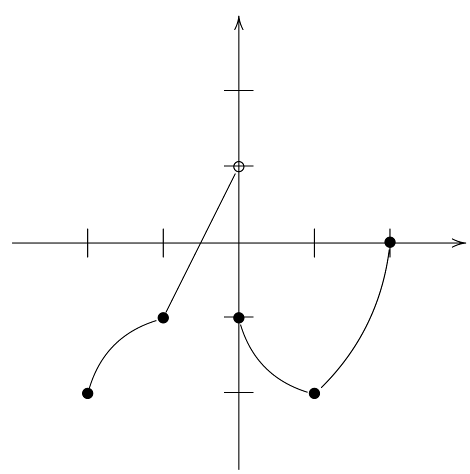
4. Determine the domain and range of each function from real numbers to real numbers:

   (a) $x \mapsto 125 - 12x$ (c) $x \mapsto (2x^3 - 7)/(3x^3 + 24)$ (c) $t \mapsto 1/(3t + 12)$

   (b) $x \mapsto (1 - x)/x$ (d) $t \mapsto 2t^2 - 10$ (d) $t \mapsto \sqrt{3t + 12}$

5. Let $f(x) = x^2 - 1$. What is $f(1)$? What is $f(\sqrt{5})$? Find numbers a and b showing that $f(a + b)$ is not the same as $f(a) + f(b)$. Also show that $f(ab) \neq f(a) f(b)$ and $f(1/a) \neq 1/f(a) \neq f(1)/f(a)$ in general. (Your examples of numbers a and b in each case might be different.)

6. The following is a graph of a function with domain $[-2, 2]$. For what x is the function positive? For what x is the function increasing? For what x is the function continuous? What is the range of the function?



7. Sketch the graphs of the following functions. Determine each function's domain and range.

(a) $a(x) = -|x-5|$ (c) $c(x) = 1/x + 3$ (e) $-\sqrt{1-(x-2)^2}$

(b) $b(x) = |-x| - 5$ (d) $d(x) = 1/(-x) + 3$ (f) $\sqrt{1-(-x)^2}$

(g) $g(x) = \begin{cases} \sqrt{1-x^2} & -1 \le x < 1 \\ 1/x & x \ge 1 \end{cases}$ (h) $h(x) = \begin{cases} |x| & x < 0 \\ x^2 & x > 0 \end{cases}$

(i) $i(x) = \begin{cases} x^3 & -1 \le x < 0 \\ \sqrt{x} & 0 \le x < 1 \\ 1/x & 1 \le x \le 3 \end{cases}$ (j) $j(x) = \begin{cases} 1/x & -1 \le x < 0 \\ \sqrt{x} & 0 \le x < 1 \\ x^3 & 1 \le x \le 3 \end{cases}$

8. For each of the following functions, compute the average rate of change between -2 and 3, between x and a, and between x and $x+h$:

(a) $x \mapsto 4$ (c) $x \mapsto -2x + 5$ (c) $t \mapsto -3/x^2$

(b) $x \mapsto 4x^2$ (d) $t \mapsto 2x^2 - x + 1$ (d) $t \mapsto 1 - x^3$

9. *(You may have done these last time already. If not, try them today.)* Let $f(x)$ is a continuous function (any function you can write down with $+$, $\times$, $-$, $\div$, $\sqrt{\cdot}$, etc., is continuous wherever it's defined). Then as x changes, the only way $f(x)$ can change from positive to negative is if it is momentarily 0 or undefined. Thus, to solve the inequality $f(x) > 0$, it's enough to find all the numbers x where $f(x) = 0$ or $f(x)$ is undefined, and then test each interval. Solve the following inequalities:

(a) $x^2 - 3x - 4 \le 0$ (c) $x^2 + 4x - 32 < 0$ (e) $\dfrac{x+1}{x+2} > \dfrac{x-3}{x+4}$ Hint: combine the fractions.

(b) $-\frac{1}{2}x^2 = \frac{7}{2}x - 5 < 0$ (d) $2x^3 - 9x^2 + 4x \ge 0$

10. (a) Let a and b be nonnegative numbers. Using the fact that a square is always nonnegative, and that squaring preserves order, show that

$$\mathrm{GM} = \sqrt{ab} \le \frac{a+b}{2} = \mathrm{AM}$$

(b) Show that if $\sqrt{ab} = (a+b)/2$, then $a = b$.

(c) Using the fact that $4 = 2 \cdot 2$, and so $\sqrt[4]{x} = \sqrt{\sqrt{x}}$, show that for any four nonnegative real numbers a, b, c, d, we have

$$\sqrt[4]{abcd} \le \frac{a+b+c+d}{4}$$

(d) Say you want to build a rectangular enclosure with 60 yards of fence. If the two sides are x and y, what is $\mathrm{AM} = (x+y)/2$? Hence, what is the maximal value for $\mathrm{GM} = \sqrt{xy}$? What is the maximal value for the area? By using 7(b), find the values of x and y that maximize the area.

(e) Say one side of your enclosure is along a perfectly straight river, and so does not need a fence. Let's let x be the length of the side parallel to the river, and y the length perpendicular to the river. What is $\mathrm{AM} = (x+2y)/2$? Write the area in terms of $\mathrm{GM} = \sqrt{2xy}$. Since the geometric mean only equals the arithmetic mean when a and b are equal, what values of x and y maximize the area?