Math 32 Discussion Problems

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In groups, solve and talk about the following problems. Be sure to take turns writing on the chalkboards. Some of these problems area easy: if everyone in your group feels very confident with a type of problem, then there's no reason to work on them, but do take the time to explain the exercises to each other. Some of these problems are hard, so you should not expect to solve all of them.

Combining and inverting functions

1. *Here are some problems from last time you may not have gotten to. Be sure to do them today if you haven't yet.* For each of the following functions, compute the average rate of change between -2 and 3, between x and a, and between x and $x+h$:

   (a) $x \mapsto 4$
   (b) $x \mapsto 4x^2$
   (c) $x \mapsto -2x+5$
   (d) $t \mapsto 2x^2 - x + 1$
   (e) $t \mapsto -3/x^2$
   (f) $t \mapsto 1 - x^3$

2. Let $f(x) = 1 - 2x^2$ and $g(x) = x + 1$. Compute

   (a) $(f+g)(x)$
   (b) $(f-g)(x)$
   (c) $(fg)(x)$
   (d) $(f/g)(x)$
   (e) $(f + 2g \cdot g)(x)$
   (f) $(g \circ f)(x)$
   (c) $(f \circ g)(x)$
   (d) $(f \circ f)(x)$
   (e) $(g \circ g)(x)$

3. A spherical weather balloon is being inflated in such a way that the radius is given by

   $$r = g(t) = \frac{1}{2}t + 2$$

   where r is in meters and t is in seconds, with $t = 0$ corresponds to the time when inflation begins. The volume of a sphere of radius r is given by

   $$V(r) = \frac{4}{3}\pi r^3.$$

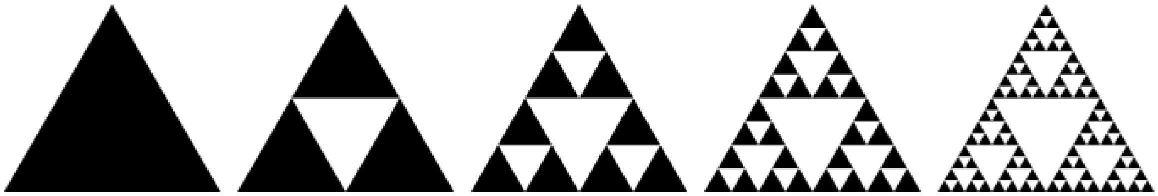
   Compute $V \circ g(t)$. Find the time at which the volume of the balloon is 36π cubic meters.

4. Let $f(x) = (3x-4)/(x-3)$. Find $f(f(113/355)$. Hint: compute $f(f(x))$ for general x first.

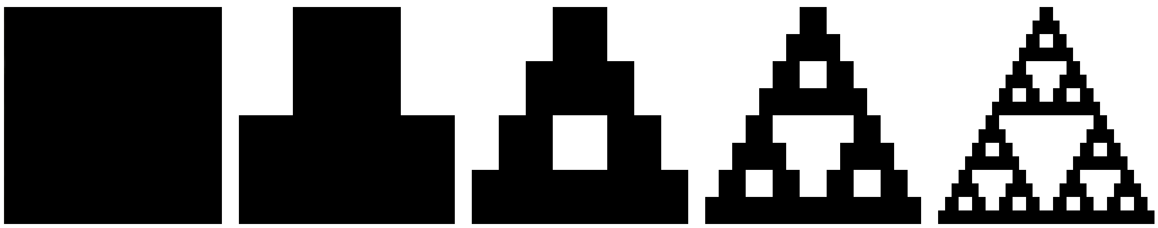
5. Consider the following functions on shapes in the plane:

   - F(shape) = the same shape, but half as big in every dimension.
   - G(shape) = three copies of the shape, placed in a triangle
   - S(shape) = $G(F$(shape))

For example, here are the first few iterates of S starting with a solid triangle (i.e. this is the sequence $\blacktriangle, S(\blacktriangle), (S \circ S)(\blacktriangle), (S \circ S \circ S)(\blacktriangle), S^{\circ 4}(\blacktriangle)$):



Here's the sequence $\blacksquare, S(\blacksquare), (S \circ S)(\blacksquare), S^{\circ 3}(\blacksquare), S^{\circ 4}(\blacksquare)$):



How does the function S affect the area of a shape? Let x be the area of a shape $\mathbf{X}$, and find a formula for the function f so that $f(x)$ is the area of $F(\mathbf{X})$. Similarly, find g so that $g(x)$ is the area of $G(\mathbf{X})$. By composing these functions, what's the function s that gives the area of $S(\mathbf{X})$ in terms of the area x of $\mathbf{X}$? What is the area of the nth iterate of S?

6. Let $f(x) = 1/(1-x)$. Show that $f \circ f$ is the inverse of f.

7. Let $f(x) = mx + b$, where $m \neq 0$.

   (a) Show that f is one-to-one, and so has an inverse.
   (b) Find a formula for $f^{-1}(x)$. Explain why it's linear.
   (c) Geometrically, the graph of the inverse of a function is its reflection across the line $x = y$. Use this to explain why f^{-1} should be linear.
   (d) What is the slope of f^{-1} when $f(x) = mx + b$? For what linear functions $f(x) = mx + b$ are the graphs of f and f^{-1} parallel? Perpendicular? Show that the graphs of $f(x)$ and $f^{-1}(-x)$ are always perpendicular.
   (e) What happens when $m = 0$?

8. Let $f(x) = 3/(x-1)$.

   (a) Find the average rate of change of f on the interval $[4, 9]$.
   (b) Find $f^{-1}(x)$, and compute the average rate of change of f^{-1} on the interval $[f(4), f(9)]$.

   How do your answers to (a) and (b) relate? By thinking geometrically, explain why this relationship will always hold for average rates of change of inverse functions.

9. Sketch a graph of each of the following functions, and use the horizontal line test to determine if it is one-to-one. If so, find its inverse:

   (a) $f(x) = -x^2 + 1$
   (b) $f(x) = -x^3 + 1$
   (c) $f(x) = \dfrac{1-3x}{2x+4}$
   (d) $f(x) = \begin{cases} x^2 & -1 \leq x \leq 0 \\ x^2 + 1 & 0 < x \end{cases}$
   (e) $f(x) = \begin{cases} x^2 & -1 \leq x < 0 \\ x^2 + 1 & 0 \leq x \end{cases}$