# Math 32 Discussion Problems

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*In groups, solve and talk about the following problems. Be sure to take turns writing on the chalkboards. Some of these problems area easy: if everyone in your group feels very confident with a type of problem, then there's no reason to work on them, but do take the time to explain the exercises to each other. Some of these problems are hard, so you should not expect to solve all of them.*

## Setting up and maximizing functions

1. For each of the following functions, state whether it makes sense to look for a highest or lowest point on the graph. Then determine the coordinate of that point.
   (a) $y = 2x^2 - 8x + 1$
   (b) $y = -3x^2 - 4x - 9$
   (c) $h = -16t^2 + 256t$
   (d) $f(x) = \sqrt{1 - (x+1)^2}$
   (e) $g(t) = 1 - \sqrt{t^2 + 1}$
   (f) $y = \left(4x^2 - 16x + 15\right)^2$

2. (a) Starting at time $t = 0$ hr, two cars move away from the same intersection. Car $A$ goes north at 40 mph, and car $B$ goes east at 30 mph. Express the distance between the two cars as a function of the time $t$.
   (b) Car $C$ passes through an intersection, heading north at 40 mph. One quarter hour later, car $D$ goes through the intersection, heading east at 30 mph. Express the distance as a function of time.
   (c) The closest cars $A$ and $B$ ever get is 0 miles, when they are both at the intersection. What is the closest cars $C$ and $D$ ever get?

3. Find two numbers adding two 20 such that the sum of their squares is as small as possible.

4. The perimeter of a rectangle is 12 m. Find the dimensions for which the diagonal is as short as possible.

5. If a ball is thrown straight up at velocity $V$, then, ignoring air resistance, its height (feet) as a function of time (seconds) will be given by $h(t) = vt - 16t^2$. Find the maximum height of the ball as a function of $V$.

6. Find the point on the curve $y = \sqrt{x}$ that is nearest to the point $(3, 0)$.

7. Suppose that $A$, $B$, and $C$ are fixed but unknown positive constants. If $x + y = C$, find the minimum value of $Ax^2 + By^2$, and find what $x$ and $y$ achieve this minimum.

8. (a) Find the coordinates of the point on the line $y = mx + b$ that is closest to the origin, in terms of $m$ and $b$.
   (b) Find the distance from that point to the origin (in terms of $m$ and $b$). This deserves to be called "the distance from the line to the origin".
   (c) Find the distance between the origin and the line $Ax + By + C = 0$ in terms of $A$, $B$, and $C$.

## The AM-GM Inequality

So far, all these problems have asked you to maximize or minimize a function that's essentially a quadratic, so you could use the vertex formula. Here's a method the works for many more functions. First, a theorem, called the "AM-GM inequality": the geometric mean of a collection of positive numbers is less than or equal to the arithmetic mean of the same collection, with equality only if all the numbers are equal. In symbols:

Let $a_1, \ldots, a_n$ be positive real numbers. Then $\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$. If $\sqrt[n]{a_1 a_2 \ldots a_n} = \frac{a_1 + a_2 + \cdots + a_n}{n}$, then $a_1 = a_2 = \cdots = a_n$. When $n = 2$, this is $\sqrt{ab} \leq \frac{a+b}{2}$, and $\sqrt{ab} = (a+b)/2$ only when $a = b$. When $n = 3$, this is $\sqrt[3]{abc} \leq \frac{a+b+c}{3}$, and $\sqrt[3]{abc} = \frac{a+b+c}{3}$ only when $a = b = c$.

The way to use the AM-GM inequality is that often if you know the sum of a bunch of numbers, then you know the maximum possible product, and if you know the product, then you know the minimum sum. For example, we can maximize $ab$ where $a + b = 3$, because then the arithmetic mean is $(a + b)/2 = 3/2$, and so $\sqrt{ab} \leq 3/2$, i.e. $ab \leq (3/2)^2$, with equality when $a = b$, which only happens with $a = b = 3/2$. Sometimes you have to do more manipulation. If you want to minimize $2a+b$ where $ab = 8$, then you use $(2a+b)/2 \geq \sqrt{(2a)b} = \sqrt{2ab} = \sqrt{16} = 4$, so $2a+b \geq 8$. Equality occurs when $2a = b$, i.e. when $8 = ab = a(2a)$, so $a = 2$ and $b = 4$.

Use the AM-GM inequality to find exact answers to the following problems:

1. Find the minimum value for the sum of a number and its reciprocal.

2. Let $xy = 8$. Find the minimum value for $x^2 + y^2$.

3. Let $xy = 1$. For every $n$ (positive or negative), find the minimum value of $x^n + y^n$.

4. Let $x + y = 3$. Find the maximum value of $x^2 y$. Hint: what does $AM - GM$ say when $n = 3$ and $a = b = x/2$ and $c = y$?

5. (This is Exercise 50 in section 4.5 from your homework.) A rectangle with fixed perimeter 36 is rotated about one of its sides, thus sweeping out a right circular cylinder. What is the maximum possible volume of that cylinder? Hint: use the previous problem.

6. (This is Exercise 47 in section 4.5 from your homework.) A rectangular can has radius $r$ and heigh $h$, and is to hold a volume $V$. Find the minimum surface area. Hint: you'll need to use the AM-GM rule with $n = 3$.

7. Find the dimensions of a rectangular box with total volume 8 and shortest main diagonal.

8. When you mail a box, the post office determines the price based on the sum *length* + *width* + *height*. Let's say you want to mail a one-foot pole. What's the minimum $l + w + h$ for a box which can hold a pole that's one foot long? Hint: $a^2 + b^2 = (a + b)^2 - 2ab$.