Math 32 Discussion Problems

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Exponential Functions

1. Use the fact that $2^{10} \approx 10^3$ to estimate 2^{60} and 4^{40}. In fact, $2^{10} = 1024$, so $2^{10}/10^3 \approx 1.02$. Use this estimate to estimate 2^{20}, 2^{30}, and 2^{60} correct to three significant figures.

2. Simplify the following expressions:
   (a) $((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}}$ (b) $(3^{2+\sqrt{5}})(3^{2-\sqrt{5}})$ (c) $10^{\pi+2}/10^{\pi-2}$ (e) $((\sqrt{5})^{3/2})^4$

3. Solve each equation:
   (a) $2^x = 32$ (b) $2^t = 1/4$ (c) $2^{3y+1} = \sqrt{2}$ (e) $8^{z+1} = 32\sqrt{2}$

4. Graph each function, marking domain, range, and asymptotes:
   (a) $y = -3^x + 3$ (b) $y = 3^{-x} - 3$ (c) $y = 2^{x-1} - 1$ (e) $y = 1 - 3^{x-1}$

5. Let $f(x) = 1 + a^x$ for some fixed constant a. Show that $\dfrac{1}{f(x)} + \dfrac{1}{f(-x)} = 1$.

6. Let $c(x) = (a^x + a^{-x})/2$ and $s(x) = (a^x - a^{-x})/2$ for some fixed constant a.
   (a) Compute $(c(x))^2 - (s(x))^2$.
   (b) Express $(c(x))^2 + (s(x))^2$ and $2\,c(x)\,s(x)$ in terms of $c(2x)$ and $s(2x)$.
   (c) Calculate $c(0)$ an $s(0)$. Determine whether each function $c(x)$ and $s(x)$ is even (symmetric across the y-axis) or odd (symmetric across the origin).

7. Given that the graph of $y = ab^x$ passes through $(0, 2)$ and $(3, 5)$, find y when $x = 6$.

8. In this problem, you'll prove that exponential functions grow faster than polynomial functions. Let $f(x) = a^x$ where $a > 1$ is a constant.
   (a) Prove that for all $x > 0$ we have $f(x) > x^0$.
   (b) Calculate the average rate of change $\dfrac{f(x+h) - f(x)}{h}$. In particular, when $h = 1$, what is $f(x+1) - f(x)$?
   (c) Thus, show that when $x > 1$, we definitely have $f(x) > (a-1)x$.
   (d) Let b be any fixed unknown number. By solving the inequality $(a-1)x > b$, find a number X (depending on a and b) so that if $x > X$ we definitely have $f(x) > b$.
   (e) Conclude that when $x > x_1$, we have $f(x) > (a-1)bx$.
   (f) In particular, when $b = 1/(a-1)$, there's a constant x_2 depending on a so that when $x > x_1$ we definitely have $f(x) > x^1$. In terms of x_1, for each positive integer n find a number x_n so that if $x > x_n$, then $f(x) > x^n$. (Hint: what happens when you take nth powers of the inequality $f(x) > x$?)
   (g) Let $p(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0$ be a polynomial. Show that when $x > 1 + |p_0| + |p_1| + \cdots + |p_n|$, then $x^{n+1} > p(x)$. Conclude that there is some number P so that if $x > P$, then $f(x) > p(x)$. Thus, exponential functions grow faster than polynomial functions.