Math 32 Discussion Problems

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/~theojf/08Fall32/

Thursday 20$^{\text{th}}$ November, 2008

Your textbook uses the notation "$\sin^{-1} x$" for the function that is the "inverse" of $\sin x$. This notation is consistent with the notation from Chapter 3, but inconsistent with the notation "$\sin^2 x$" for $(\sin x)^2$. A less confusing is "$\arcsin x$" for the inverse function of $\sin x$; of course, $\csc x = (\sin x)^{-1}$.

Trigonometric Equations and Inverse Functions

1. Let $\tan^{-1} = \arctan$ be the inverse-tangent function, defined as outputting the unique $y \in [-\pi/2, \pi/2]$ such that $x = \tan y$.

   (a) What is $\tan^{-1} 0$? What is $\tan(\arctan 3\pi)$?

   (b) What is the domain of $\tan^{-1}$? What is the range?

   (c) Graph $\tan^{-1}(\tan x)$ and $\tan(\tan^{-1} x)$.

   (d) Show that $\cot(\tan^{-1} x) = 1/x$ for every $x \neq 0$.

2. (a) Show that if $x \in [-1, 1]$, then $\cos(\sin^{-1} x) = \sqrt{1 - x^2} = \sin(\cos^{-1} x)$.

   (b) Evaluate $\tan(\sin^{-1} x)$ and $\tan(\cos^{-1} x)$.

   (c) Find $\sin(\tan^{-1} x)$ and $\cos(\tan^{-1} x)$.

3. When we defined the inverse sine, cosine, and tangent) functions, we had to limit the domains of sine, cosine, and tangent. There was always a best choice for how to limit the domain: we always chose the largest continuous domain of sine, cosine, or tangent that contained 0 on which the function was one-to-one.

   (a) Does this method work to define the secant-inverse function $\sec^{-1}$? Graph $y = f(x) = \cos^{-1}(1/x)$ and specify the domain and range. Prove that $f(x)$ is an inverse function to $\sec(x)$, by calculating $f(\sec(x))$ and $\sec(f(x))$.

   (b) Does this method work to define the cosecant-inverse function $\csc^{-1}$? Graph $y = g(x) = \csc^{-1}(1/x)$ and specify the domain and range. Prove that $g(x)$ is an inverse function to $\csc(x)$, by calculating $g(\csc(x))$ and $\csc(g(x))$.

4. Determine all solutions to the given equations. Express your answers using radian measure:

   (a) $\sin\theta = \sqrt{2}/2$ (d) $\tan\theta + \sqrt{3}/3 = 0$ (g) $\cos\theta + 2\sec\theta = -3$

   (b) $\sin\theta + \sqrt{2}/2 = 0$ (e) $2\sin^2 x - 3\sin x + 1 = 0$ (h) $2\cot^2 x + \csc^2 x - 2 = 0$

   (c) $\cos\theta = 1/2$ (f) $\sin^2 x - \sin x - 6 = 0$ (i) $\sec\alpha + \tan\alpha = \sqrt{3}$

5. Determine all solutions to the given equations:

   (a) $\tan 2\theta = -1$ (b) $\sin(\theta/2) = 1/2$ (c) $2\sin^2\theta = \cos 2\theta$ (d) $\sin\theta = \cos(\theta/2)$

6. **This problem involves complex numbers, which we haven't talked about yet. Do at your own risk.**

   Let's define a new number, called i, which has the property that $i^2 = -1$. Of course, i cannot be a real number, as the squares of real numbers are not negative. We won't impose any other conditions on i. So our extended number system contains all the real numbers $\mathbb{R}$, and also the number i. This extended number system is call the Complex Numbers, written $\mathbb{C}$.

   If b is a real number, then bi will be in the complex numbers. Real multiples of i are called "pure imaginary" numbers. If a is another real number, then $a + bi$ is as well. But every number is a real number plus a pure imaginary number. Here's why:

   (a) If a, b, c, and d are real numbers, show by factoring that

   $$(a + bi) + (c + di) = (a + c) + (b + d)i$$

   (b) If a, b, c, and d are real numbers, use the fact that $i^2 = -1$ to show that

   $$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

   Thus any number you can get by adding and multiplying numbers of the form Real + Pure Imaginary can again be written in that form. (Division, square roots, etc. also all stay in this form, but proving that would take us too far afield from the topic at hand.)

   How does this relate to trigonometry? Let's define a function from the real numbers to the complex numbers by the following formula:

   $$f(x) = \cos(x) + \sin(x)\, i$$

   (c) Using problem (b) and the sum-of-angles formula to show that

   $$f(x + y) = f(x)\, f(y)$$

   (d) Use part (c) to show that

   $$f(3x) = f(x)^3$$

   and more generally that $f(nx) = f(x)^n$ for any positive integer n.

   (e) Expand out the formula in part (d) in terms of sines and cosines. I.e. expand out

   $$\cos(3x) + \sin(3x)\, i = (\cos(x) + \sin(x)\, i)^3$$

   You should use the fact that $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$. What is i^3?

   (f) It turns out that two complex numbers $a+bi$ and $c+di$ are equal if and only if their real parts are equal ($a = c$) and their pure-imaginary parts are equal ($b = d$). By comparing just the real parts or just the pure imaginary parts of your answer to problem (e), conclude formulas for $\cos(3x)$ and $\sin(3x)$.

   (g) Use the fact that

   $$(A + B)^7 = A^7 + 7A^6B + 21A^5B^2 + 35A^4B^3 + 35A^3B^4 + 21A^2B^5 + 7AB^6 + B^7$$

   to derive formulas for $\cos(7x)$ and $\sin(7x)$.