Math 32 Discussion Problems

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Inequalities and systems thereof

1. Graph the solution sets to the following systems of inequalities. Specify all vertices of the regions.

   (a) $\begin{cases} y < x \\ x^2 + y^2 < 1 \end{cases}$

   (b) $\begin{cases} x \geq 0 \\ y \geq 0 \\ y < \sqrt{x} \\ x \leq 4 \end{cases}$

   (c) $\begin{cases} x \geq 0 \\ y \geq e^x \\ y \leq e^{-1} + 1 \end{cases}$

   (d) $\begin{cases} y \geq x + 5 \\ y \geq -2x + 14 \\ x \geq 0 \\ y \geq 0 \end{cases}$

   (e) $\begin{cases} y \geq 2x \\ y \geq -x + 6 \end{cases}$

   (f) $\begin{cases} 2x - y + 8 \geq 0 \\ x + 3y \leq 23 \\ 5x + y \leq 45 \\ x \geq 0 \\ y \geq 0 \end{cases}$

2. For each of the following functions $f(x)$, graph it, and use the graph to solve the inequality $f(x) \geq 0$:

   (a) $f(x) = x^2 - 3x - 4$

   (b) $f(x) = -\frac{1}{2}x^2 - \frac{7}{2}x - 5$

   (c) $f(x) = -3(x+3)^2$

   (d) $f(x) = \ln(x+1)$

   (e) $f(x) = -1 + \sqrt{x}$

   (f) $f(x) = -1 + \sqrt{4 - x^2}$

3. It is a basic theorem from calculus that a continuous function cannot change sign except at points where it is 0 or undefined. Such points are called the *key numbers* of the function. Thus, the solution to any inequality of the form $f(x) \geq 0$ (or $f(x) \leq 0$ or $f(x) > 0$ or $f(x) < 0$) is a disjoint union of intervals whose endpoints are key numbers. Remember that the product or ratio of continuous functions is 0 or undefined exactly when one of the terms is 0 or undefined. Solve the following inequalities:

   (a) $(x+4)(x+5)(x+6) < 0$

   (b) $(x-2)^2(3x+1)^3(3x-1) > 0$

   (c) $x^4 - 25x^2 + 144 \leq 0$

   (d) $(x+4)/(2x-5) \leq 0$

   (e) $(x^2-1)/(x^2+8x+15) \geq 0$

   (f) $(x^2-3x+1)/(1-x) < 0$

4. The fact that $f(x)\,g(x) = 0$ only if $f(x) = 0$ or $g(x) = 0$ is a special property of the number 0. To solve more complicated inequalities using key numbers, you first have to rearrange the inequality into the form function ≥ 0 (or ≤ 0, etc.). Solve the following inequalities:

(a) $\dfrac{2x}{x-2} < 3$

(b) $\dfrac{2}{x} < \dfrac{x}{2}$

(c) $\dfrac{2x}{x+5} + \dfrac{x-1}{x-5} < \dfrac{1}{5}$

(d) $\dfrac{x+1}{x+2} > \dfrac{x-3}{x+4}$

5. Sometimes it can be more efficient to just start solving an inequality rather than always pigeon-holing into the key numbers method. But remember that algebraic manipulation can change an inequality. For example, if $a < b$, then sometimes $\frac{1}{a} < \frac{1}{b}$ and sometimes $\frac{1}{a} > \frac{1}{b}$: the inequality stays the same if a is negative and b is positive, and switches direction if a and b have the same sign.

   You can remember whether an operation $x \mapsto f(x)$ changes the direction of an inequality by graphing $y = f(x)$: if the graph is upward-sloping, f preserves inequalities, and if the graph is downward-sloping, f reverses inequalities. For example, exponentiation and logarithms preserve inequalities, but multiplying by a negative number reverses them.

   Don't forget to check that any proposed solution to an inequality or equality is in the domain of your original functions. Solve the following inequalities:

   (a) $\dfrac{2}{3}\left(1 - e^{-x}\right) < 1$

   (b) $3\log_{10}(4x+3) < 1$

   (c) $4^{5-x} > 15$

   (d) $\log_2 x \geq 0$

   (e) $\ln\left(\dfrac{3x-2}{4x+1}\right) > \ln 4$

   (f) $10^{-x^2} \leq 10^{-12}$

   (g) $e^{1/(x-1)} > 1$

   (h) $e^{(1/x)-1} > 1$

6. My last bit of advice is the repeated reminder that graphing is a good way to understand functions. I find this is especially true for solving inequalities with trigonometric functions. Key numbers and algebraic manipulation are also important techniques. Solve the following inequalities:

   (a) $\sin\theta > \sqrt{3}/2$

   (b) $\sin\theta \leq -1/2$

   (c) $\cos\theta > -1$

   (d) $\tan\theta \geq \sqrt{3}$

   (e) $\tan x < 0$

   (f) $2\cos^2\theta + \cos\theta > 0$

   (g) $\cos^2 t\,\sin t - \sin t \geq 0$

   (h) $2\cos^2 x - \sin x - 1 \leq 0$

   (i) $\sqrt{3}\sin t - \sqrt{1+\sin^2 t} > 0$

   (j) $\sec\alpha + \tan\alpha < 0$