Math 32 Discussion Problems

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Polynomials

1. Use the quadratic formula (and factoring by grouping) to find all complex solutions to the given equations:

   (a) $x^2 - 6x + 12 = 0$

   (b) $-10z^2 + 4z - 2 = 0$

   (c) $3z^2 - 7z + 5 = 0$

   (d) $\frac{1}{2}z^2 + 2z + \frac{9}{4} = 0$

   (e) $2x^3 + 4x^2 + 3x + 6 = 0$

   (f) $x^5 + 4x^3 + 8x^2 + 32 = 0$

2. *This problem explains one of the deep connections between complex numbers and trigonometry. You may have done it already; it was at the end of the handout from November 20.*

   Let's define a function from the real numbers to the complex numbers by the following formula:

   $$f(x) = \cos(x) + \sin(x)\, i$$

   (a) Using and the sum-of-angles formula to show that $f(x+y) = f(x)\, f(y)$.

   (b) Use part (a) to show that $f(3x) = f(x)^3$. More generally, show that $f(nx) = f(x)^n$ for any positive integer n.

   (c) Expand out the formula in part (b) in terms of sines and cosines. I.e. expand out

   $$\cos(3x) + \sin(3x)\, i = (\cos(x) + \sin(x)\, i)^3$$

   You should use the fact that $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$. What is i^3?

   (d) By comparing just the real parts or just the pure imaginary parts of your answer to problem (c), conclude formulas for $\cos(3x)$ and $\sin(3x)$.

   (e) Use the fact that

   $$(A+B)^7 = A^7 + 7A^6B + 21A^5B^2 + 35A^4B^3 + 35A^3B^4 + 21A^2B^5 + 7AB^6 + B^7$$

   to derive formulas for $\cos(7x)$ and $\sin(7x)$.

3. Divide the following polynomials. Do these enough to get the hang of it (maybe you already have the hang of it from your homework.) Be sure to do the last four.

   (a) $\dfrac{x^4 - 4x^3 + 6x^2 - 4x + 1}{x - 1}$

   (b) $\dfrac{4x^3 - x^2 + 8x - 1}{x^2 - x + 1}$

   (c) $\dfrac{x^6 + 64}{x - 2}$

   (d) $\dfrac{8x^6 - 36x^4 + 54x^2 - 27}{2x^2 - 3}$

   (e) $\dfrac{4y^4 - y^3 + 2y - 1}{2y^2 - 3y - 4}$

   (f) $\dfrac{2t^5 - 6t^4 - t^2 + 2t + 3}{t^3 - 2}$

   (g) $\dfrac{1 + z + z^2 + z^3}{1 + z + z^2}$

   (h) $\dfrac{ax^3 + bx^2 + cx + d}{x - r}$

   (i) $\dfrac{x^2 - 4x + i}{x - i}$

   (j) $\dfrac{x^2 - 2x + 2}{x - (1 + i)}$

   (k) $\dfrac{x^3 - 2x^2 - 4}{x - 3i}$

   (l) $\dfrac{x^3 - x^2 + 4x - 4}{x + 2i}$

4. Find the value of k such that when $x^3 + kx + 1$ is divided by $x + 3$, the remainder is -4.

5. For each of the following polynomials, I've listed one or more roots. Find the rest of the roots. (Hint: use synthetic division.)

   (a) $x^3 - 4x^2 - 9x + 36$; -3 is a root.

   (b) $x^3 + 7x^2 + 11x + 5$; -1 is a root.

   (c) $x^3 + x^2 - 7x + 5$; 1 is a root.

   (d) $3x^3 - 5x^2 - 16x + 12$; -2 is a root.

   (e) $2x^3 - 5x^2 - 46x + 24$; 6 is a root.

   (f) $2x^3 + x^2 - 5x - 3$; $-3/2$ is a root.

   (g) $x^4 - 15x^3 + 75x^2 - 125x$; 5 is a root.

   (h) $x^4 + 2x^3 - 23x^2 - 24x + 144$; -4 and 3 are roots.

   (i) $6x^5 + 5x^4 - 29x^3 - 25x^2 - 5x$; $\sqrt{5}$ and $-1/3$ are roots.

6. Find a polynomial of degree 3 ...

   (a) ... such that the coefficient of x^3 is 1 and the roots are 3, -4, and 5.

   (b) ... with integer coefficients such that the roots are $1/2$, $2/5$, and $-3/4$.

   (c) ... with a root of multiplicity two at -1 and such that $x + 6$ is a factor.

7. Find all possible values of b such that one root of the equation $x^2 + bx + 1 = 0$ is twice the other one.

8. Let the roots of a polynomial be: $\sqrt{3}$ with mulitplicity 2, $-\sqrt{3}$ with multiplicity 2, $4i$ with multiplicity 1, and $-4i$ with multiplicity 1. If the leading coefficient is 1, what is the polynomial?

9. Factor $x^4 + 64$ into linear factors.

10. Define the nth *quantum integer* to be the polynomial $f_n(q) = 1 + q + \cdots + q^{n-1}$. For example, $f_1(q) = 1$, $f_2(q) = 1 + q$, and $f_3(q) = 1 + q + q^2$. (The conventional notation is to write $[n]$ for f_n, but I think this might be confusing.)

    (a) What is $f_n(1)$? What is $f_n(0)$? What is $f_n(-1)$? What is $f_n(i)$?

    (b) What is $f_n(q) \times f_2(-q)$?

    (c) Use synthetic division to show that $f_n(q)$ is divisible by $f_2(q)$ if and only if n is even.

    (d) More generally, show that $f_n(q)$ is divisible by $f_d(q)$ if and only if n is divisible by d.

    (e) If d divides n, show that $f_n(q)/f_d(q) = f_{n/d}(q^d)$.

    (f) Completely factor $f_{12}(q) = 1 + q + q^2 + \cdots + q^{10} + q^{11}$.

11. (a) Show that $(x - r_1)(x - r_2)\dots(x - r_n) = x^n - (r_1 + r_2 + \cdots + r_n)x^{n-1} +$ other terms, for any integer n and numbers $r_1, \dots, r_n$.

    (b) Let r_1, r_2, r_3, and r_4 be four real roots of the equation $x^4 + ax^2 + bx + c = 0$. Show that $r_1 + r_2 + r_3 + r_4 = 0$. Hint: explain how to factor $x^4 + ax^2 + bx + c$ into linears.

    (c) Suppose that a circle intersects the parabola $y = x^2$ in the points (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4). Show that $x_1 + x_2 + x_3 + x_4 = 0$.