Math 32 Quiz

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Name: _____ Score: /10

You have twenty minutes to complete this quiz. You may not use calculators or notes, but the chalkboards are yours.

1. (4 pts) Use Gaussian elimination to find all solutions to the following system of equations:

$$\begin{cases} x + 4y - z = 0 \\ 3x + y - z = -1 \\ 4x - 4y + 5z = -7 \end{cases}$$

We replace E_2 *by* $E_2 - 3E_1$ *and* E_3 *by* $E_3 - 4E_1$*:*

$$\begin{cases} x + 4y - z = 0 \\ -11y + 2z = -1 \\ -20y + 9z = -7 \end{cases}$$

Now we could divide E_2 *by* -11*, but here's a trick: We replace* E_3 *be* $E_3 - 2E_2$*:*

$$\begin{cases} x + 4y - z = 0 \\ -11y + 2z = -1 \\ 2y + 5z = -5 \end{cases}$$

Now we replace E_2 *by* $E_2 + 5E_3$*:*

$$\begin{cases} x + 4y - z = 0 \\ -y + 27z = -26 \\ 2y + 5z = -5 \end{cases}$$

Now we replace E_3 *by* $E_3 + 2E_2$ *to cancel the* y*:*

$$\begin{cases} x + 4y - z = 0 \\ -y + 22z = -26 \\ 59z = -57 \end{cases}$$

Hence we can divide the last line by 59 *and get* $z = -57/59$*. Thus we can substitute back in to get the other variables:*

$$\begin{cases} x + 4y - \quad z = 0 \\ \quad - \quad y + 22z = -26 \\ \qquad\qquad z = -57/59 \end{cases}$$

$$\begin{cases} x + 4y \quad\quad = -1 \\ \quad - \quad y \quad\quad = -26 + 22 \cdot 57/59 \\ \qquad\qquad\quad = -1512/59 \\ \qquad\quad z = -57/59 \end{cases}$$

$$\begin{cases} x + 4y \quad\quad = -6107/59 \\ \qquad y \quad\quad = 1512/59 \\ \qquad\quad z = -57/59 \end{cases}$$

$$\begin{cases} x = -6107/59 \\ y = 1512/59 \\ z = -57/49 \end{cases}$$

Thus, the final answer is $\boxed{(x, y, z) = \dfrac{1}{59}(-6107, 1512, -57)}$.

2. (3 pts) Find all solutions to the following system of two equations and three unknowns:

$$\begin{cases} x + 2y + 3z = 1 \\ 2x - y - z = 2 \end{cases}$$

We subtract two times the first row from the second and get $-5y - 7z = 0$. *Thus* $y = -\frac{7}{5}z$. *Substituting into the first equation gives*

$$1 = x + 2\left(-\frac{7}{5}z\right) + 3z = x + \frac{1}{5}z$$

Thus for any z, $\boxed{(x, y, z) = \left(1 - \frac{1}{5}z, -\frac{7}{5}z, z\right)}$ *is a solution.*

3. (3 pts) There are four solutions to the following system of equations. Find them.

$$\begin{cases} x^2 + y^2 = 17 \\ xy = 4 \end{cases}$$

There are various ways to go about this problem. We can solve the second equation for y

$$y = \frac{4}{x}$$

and substitute into the first equation

$$x^2 + \frac{16}{x^2} = 17$$

Multiply both sides by x^2, and solve the resulting equation as a quadratic in x^2:

$$\begin{aligned} (x^2)^2 - 17x^2 + 16 &= 0 \\ (x^2 - 16)(x^2 - 1) &= 0 \\ x^2 &= 16 \text{ or } 1 \\ x &= \pm 4 \text{ or } \pm 1 \\ (x, y) &= \boxed{(4,1),\ (-4,-1),\ (1,4),\ \text{or } (-1,-4)} \end{aligned}$$

Here's an alternate trick. By adding or subtracting twice the second equation to the first, we get perfect squares:

$$\begin{aligned} (x+y)^2 = x^2 + y^2 + 2xy &= 17 + 8 = 25 \\ (x-y)^2 = x^2 + y^2 - 2xy &= 17 - 8 = 9 \\ x + y &= \pm 5 \\ x - y &= \pm 3 \end{aligned}$$

Thus we get two numbers that add to ± 5 and differ by 3, so the numbers are $\pm(1,4)$ in either order.