Math 53 Quiz 10

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Name: ____

Time (circle one): 12:10 - 1:00 3:10 - 4:00

Please use extra paper as necessary. For each part, partial credit will be assigned based on correct work (you do need to show some work, enough so that I know how you solved the problem). Please simplify and box *your answers.*

a. (3 pts) Given a continuously twice-differentiable (scalar) function f, prove the following product rule:

$$\vec{\nabla} \cdot \left(f\, \vec{\nabla} f \right) = \left(\vec{\nabla} f \right)^2 + f\, \nabla^2 f$$

In general, for a vector field $\vec{v}$, we have the product rule $\vec{\nabla} \cdot (f\vec{v}) = \vec{\nabla} f \cdot \vec{v} + f\, \vec{\nabla} \cdot \vec{v}$:

$$\begin{aligned} \vec{\nabla} \cdot (f\vec{v}) &= \frac{\partial}{\partial x}\left[f v_1 \right] + \dots \\ &= \left(\frac{\partial f}{\partial x} v_1 + f \frac{\partial v_1}{\partial x} \right) + \dots \\ &= \left(\frac{\partial f}{\partial x} v_1 + \dots \right) + \left(f \frac{\partial v_1}{\partial x} + \dots \right) \\ &= \vec{\nabla} f \cdot \vec{v} + f\, \vec{\nabla} \cdot \vec{v} \end{aligned}$$

Then

$$\vec{\nabla} \cdot \left(f\, \vec{\nabla} f \right) = \vec{\nabla} f \cdot \vec{\nabla} f + f \vec{\nabla} \cdot \vec{\nabla} f = \left(\vec{\nabla} f \right)^2 + f\, \nabla^2 f$$

b. (3 pts) Hence, for a region R (in the plane) with boundary ∂R (and where $\vec{n}$ is a unit vector perpendicular to the boundary, ds is an infinitesimal piece of boundary, dA is the area form on R, etc.), on which f is defined and continuously twice-differentiable, prove:

$$\oint_{\partial R} f\, \vec{\nabla} f\, \vec{n} ds = \iint_R (\vec{\nabla} f)^2\, dA + \iint_R f\, \nabla^2 f\, dA$$

Applying the above product rule and the second version of Green's theorem (i.e. the Divergence Theorem):

$$\begin{aligned}\oint_{\partial R} f\,\vec{\nabla} f\,\vec{n} ds &= \iint_R \vec{\nabla}\cdot\left(f\,\vec{\nabla} f\right) dA \\ &= \iint_R \left(\left(\vec{\nabla} f\right)^2 + f\,\nabla^2 f\right) dA \\ &= \iint_R (\vec{\nabla} f)^2\,dA + \iint_R f\,\nabla^2 f\,dA\end{aligned}$$

c. (2 pts) Now assume that f is a harmonic function, i.e. $\nabla^2 f = 0$, and that f is 0 on the boundary ∂R. Then what can you conclude about the value of $\vec{\nabla} f$ inside the region R? (Hint: The integral of a positive function is positive.)

If f is 0 on the boundary, then

$$\oint_{\partial R} f\,\vec{\nabla} f\,\vec{n} ds = \oint_{\partial R} 0\,\vec{\nabla} f\,\vec{n} ds = 0$$

If f is harmonic, then

$$\iint_R f\,\nabla^2 f\,dA = \iint_R f\,0\,dA = 0$$

Hence by the previous problem

$$\iint_R (\vec{\nabla} f)^2\,dA = 0$$

But $(\vec{\nabla} f)^2 \geq 0$, so the only way the integral can be zero is if $\vec{\nabla} f$ is identically 0 on R.

d. (2 pts) Hence, what are all possible harmonic functions f on a region that vanish on the boundary of that region? (Hint: What do you know about functions with zero derivative?)

Since $\vec{\nabla} f = 0$, we know that f is constant. Since f is 0 on the boundary, we know that it must be 0 anywhere. Hence $\boxed{f(x, y) = 0}$ is the only harmonic function on a region that vanishes on the boundary of a region.