Math 53 Quiz 12

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Name: ____

Time (circle one): 12:10 - 1:00 3:10 - 4:00

Please use extra paper as necessary. For each part, partial credit will be assigned based on correct work (you do need to show some work, enough so that I know how you solved the problem). Please simplify and box *your answers.*

When a solid body is partially submerged in liquid, it feels a water-pressure force on each point of its surface. If the water pressure at a point (x, y, z) is $P(x, y, z)$, then the force on a small area dS with outward normal $\hat{\mathbf{n}}$ of the surface is:

$$\frac{d\vec{\mathbf{F}}}{dS} = -P\,\hat{\mathbf{n}}$$

(The minus sign is because the pressure points in.) The aggregate of all the little bits of pressure can act as a buoyancy force; the total force is:

$$\vec{\mathbf{F}}_{\text{total}} = -\iint_S P\,\hat{\mathbf{n}}\,dS$$

where S is the surface of the object under the water. The component of $\vec{\mathbf{F}}$ in the $\hat{\mathbf{i}}$-direction (or $\hat{\mathbf{j}}$ or $\hat{\mathbf{k}}$ or anything else) is:

$$F_{\hat{\mathbf{i}}} = \hat{\mathbf{i}} \cdot \vec{\mathbf{F}} = \iint_S -P\hat{\mathbf{i}} \cdot \hat{\mathbf{n}}\,dS$$

In general, the pressure is

$$P(x, y, z) = \begin{cases} -\rho g z, & z \le 0 \\ 0, & z \ge 0 \end{cases}$$

where ρ and g are constants (physically, ρ is the density of water, and g is the strength of gravity); the minus sign is because we measure z going up, but pressure is higher as you go down. We set $P(x, y, z) = 0$ for $z \ge 0$, thinking of the water level as $z = 0$, and pretending that there is no air pressure.

a. (3 pts) Is $-P\hat{\mathbf{i}} = \rho g z\hat{\mathbf{i}}$ the curl of a vector field? If not, explain why not. If so, find a vector field $\vec{\mathbf{v}}$ so that $\vec{\nabla} \times \vec{\mathbf{v}} = -P\hat{\mathbf{i}}$.

We take the divergence:

$$\vec{\nabla} \cdot (-P\hat{\mathbf{i}}) = \frac{\partial}{\partial x}(\rho g z) + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}0 = 0$$

so, $\boxed{\textit{yes}}$*, this is a curl (since the domain of P is contractible). Guessing that $-P\hat{\mathbf{i}}$ has an antiderivative of the form $Q\hat{\mathbf{j}}$, we see that*

$$\frac{\partial Q}{\partial z} = -\rho g z \qquad \frac{\partial Q}{\partial x} = 0$$

and hence one possible answers is

$$Q = -\rho g z^2/2 \qquad \boxed{\vec{\mathbf{v}} = -\frac{\rho g z^2}{2}\hat{\mathbf{j}}}$$

b. (2 pts) Consider the situation when the object is only partially submerged, as in Figure 1. Let S be the surface below the level of the water, and V the volume of the object below the water. Calculuate, in terms of V, ρ, and g, the integral:

$$F_{\hat{\mathbf{i}}} = \iint_S -P\hat{\mathbf{i}} \cdot \hat{\mathbf{n}}\, dS = \iint_S \rho g z\hat{\mathbf{i}} \cdot \hat{\mathbf{n}}\, dS$$

We use Stokes' Theorem:

$$\iint_S \rho g z\hat{\mathbf{i}} \cdot \hat{\mathbf{n}}\, dS = \oint_{\partial S} -\frac{\rho g z^2}{2}\hat{\mathbf{j}} \cdot d\vec{\mathbf{r}} = -\frac{\rho g}{2}\oint_{\partial S} z^2\, dy$$

But the boundary ∂S is at the water's surface: it is contained within the plane $\{z = 0\}$. Hence the integral is an integral of 0 over a path, so is $\boxed{0}$*. Hence the floating body does not feel any net force in the $\hat{\mathbf{i}}$ direction.*

c. (3 pts) Is $-P\hat{\mathbf{k}} = \rho g z\hat{\mathbf{k}}$ the curl of a vector field? If not, explain why not. If so, find a vector field $\vec{\mathbf{v}}$ so that $\vec{\nabla} \times \vec{\mathbf{v}} = -P\hat{\mathbf{k}}$.

We take the divergence:

$$\vec{\nabla} \cdot (-P\hat{\mathbf{k}}) = \frac{\partial}{\partial x}0 + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}(\rho g z) = \rho g \neq 0$$

Since this is not 0, $\boxed{-P\hat{\mathbf{k}} \textit{ is not a curl}}$*; the divergence of a curl is always 0.*

d. (2 pts) Consider the situation when the object is fully submerged, as in Figure 2. Let S be the surface below the level of the water, and V the volume of the object below the water. Calculuate, in terms of V, ρ, and g, the integral:

$$F_{\hat{\mathbf{k}}} = \iint_S -P\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\, dS = \iint_S \rho g z\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\, dS$$

We use the divergence theorem:

$$
\begin{aligned}
\iint_S \rho g z \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \, dS &= \iiint\limits_{\textit{interior of } S} \vec{\nabla} \cdot (\rho g z \hat{\mathbf{k}}) \, dV \\
&= \iiint\limits_{\textit{interior of } S} \rho g \, dV \\
&= \rho g \iiint\limits_{\textit{interior of } S} dV \\
&= \boxed{\rho g V}
\end{aligned}
$$

Hence the submerged object feels an upward force equal to the weight of the displaced water.