Math 53 Quiz 7

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Name: ____

Time (circle one): 12:10 - 1:00 3:10 - 4:00

By squaring and switching to polar coordinates, evaluate the following improper integral:

$$\int_0^\infty \left(\frac{1}{2}\right)^{x^2} dx$$

You should use the ideas and methods from your homework (not just the result).

Please use extra paper as necessary. For each part, partial credit will be assigned based on correct work (you do need to show some work, enough so that I know how you solved the problem). Please simplify and $\boxed{\text{box}}$ your answers.

We integrate over the first quadrant; in polar coordinates, this is $0 \le r < \infty$ *and* $0 \le \theta \le \pi/2$. *We substitute* $u = r^2$, *the integrate using* $\int a^u\,du = a^u/\ln(a)$.

$$\begin{aligned}
\left(\int_0^\infty \left(\frac{1}{2}\right)^{x^2} dx\right)^2 &= \int_0^\infty \int_0^\infty \left(\frac{1}{2}\right)^{x^2} \left(\frac{1}{2}\right)^{y^2} dy\,dx = \iint\limits_{\substack{\text{first}\\ \text{quadrant}}} \left(\frac{1}{2}\right)^{x^2+y^2} dA \\
&= \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} \left(\frac{1}{2}\right)^{r^2} r\,dr\,d\theta \\
&= \frac{\pi}{2} \int_{u=0}^{\infty} \left(\frac{1}{2}\right)^{u} \frac{du}{2} \\
&= \frac{\pi}{4} \left[\frac{1}{\ln(1/2)} \left(\frac{1}{2}\right)^{u}\right]_{u=0}^{\infty} = \frac{\pi}{4} \frac{1}{-\ln 2} \left[\left(\frac{1}{2}\right)^{\infty} - \left(\frac{1}{2}\right)^{0}\right] \\
&= -\frac{\pi}{4\ln 2}\,[0-1] = \frac{\pi}{4\ln 2}
\end{aligned}$$

Thus, $$\int_0^\infty \left(\frac{1}{2}\right)^{x^2} dx = \boxed{\frac{1}{2}\sqrt{\frac{\pi}{\ln 2}}}$$