Math 53 Quiz 9

18 April 2008

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Name: ____

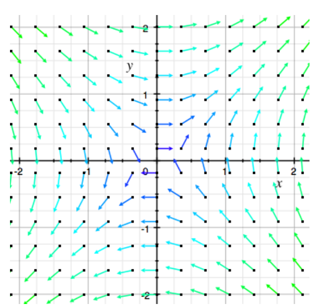
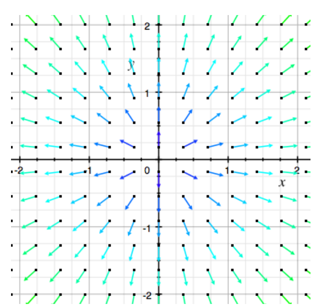
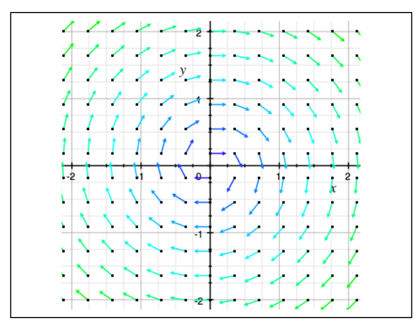
Time (circle one): 12:10 - 1:00 3:10 - 4:00

Please use extra paper as necessary. For each part, partial credit will be assigned based on correct work (you do need to show some work, enough so that I know how you solved the problem). Please simplify and box *your answers.*

For this quiz, consider the vector field:

$$\vec{v}(x,y) = y\vec{\imath} - x\vec{\jmath}$$

a. (1 pt) Which of the following is a graph of $\vec{v}$?

(Note: My graphing calculator normalizes each vector in a vector field to a unit vector, and displays the magnitude as a color. But this is a black-and-white printer. So you'll have to make your pick based only on the direction of the vectors.)

b. (2 pt) Is there a function $F(x,y)$ so that $\vec{v}(x,y) = \vec{\nabla} F$? If so, find such a function. If not, explain why not.

We let $\vec{v} = P\vec{\imath} + Q\vec{\jmath}$ *and compute* $\partial Q/\partial x - \partial P/\partial y = -1 - 1 = -2 \neq 0$. *Hence there is no function* F *so that* $\vec{v} = \vec{\nabla} F = (\partial F/\partial x)\vec{\imath} + (\partial F/\partial y)\vec{\jmath}$ *— if there were, then the two "mixed" partials would agree.*

c. (4 pt) Given real numbers a, b, evaluate $\int_\gamma \vec{v} \cdot d\vec{r}$ along the following paths:

- $\vec{r}(t) = (at, 0)$ for t ranging from 0 to 1.

$$\int_{t=0}^{t} \langle 0, -at\rangle \cdot \langle a, 0\rangle dt = \int_{t=0}^{1} 0\, dt = \boxed{0}$$

- $\vec{r}(t) = (0, b - bt)$ for t ranging from 0 to 1.

$$\int_{t=0}^{t} \langle b - bt, 0\rangle \cdot \langle 0, -b\rangle dt = \int_{t=0}^{1} 0\, dt = \boxed{0}$$

- $\vec{r}(t) = (a - at, bt)$ for t ranging from 0 to 1.

$$\int_{t=0}^{t} \langle bt, at - a\rangle \cdot \langle -a, b\rangle dt = \int_{t=0}^{1} (-abt + abt - ab) dt = \boxed{-ab}$$

d. (3 pt) What is the relationship between the sum of the three answers in part c. and the area of the triangle with corners $(0, 0)$, $(a, 0)$, and $(0, b)$? Use Green's Theorem to prove that this relationship holds for path integrals around the boundary of any region. (Hint: This is a good chance to check your answers to part c., if Green's theorem suggests a different relationship between the area of a region and the integral of $\vec{v}$ around the boundary of that region.)

The area of the triangle is $ab/2$, and our answer is -2 times that number. The sum of the three answers is the integral around the perimeter of the triangle in the counterclockwise direction. For any path integral around the boundary ∂R of a region R, we have via Green's Theorem:

$$\oint_{\partial R} \vec{v} \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \iint_R (-2)\, dx\, dy = -2 \times \iint_R dA = 2 \times \textit{area of } R$$