math 53 may 12. 2008

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① Remember that $\sin^2 + \cos^2 = -\sinh^2 + \cosh^2 = 1$. Find a parameterization for the ~~conic~~ conic surface

$$\{Ax^2 + By^2 + Cz^2 = 1\}, \qquad A, B, C \neq 0$$

How does your answer depend on the signs of A, B, and C?

② Let $A, B, C > 0$. Use your parameterization to write out, but do not integrate,

$$\oiint_S \vec{F} \cdot \hat{n}\, dS$$

where S is the level set $S = \{(x,y,z) : Ax^2 + By^2 + Cz^2 = 1\}$ and $\vec{F}$ is the vector field $\cancel{ax^2\hat{\imath} + by^2\hat{\jmath} + cz^2}\hat{k}$.

$$\vec{F} = ax\,\hat{\imath} + by\,\hat{\jmath} + cz\,\hat{k}$$

Use the divergence theorem to evaluate the integral.

③ What condition on a, b, c makes $\vec{F}$ the curl of a vector field? If a, b, c satisfy that condition, find $\vec{G}$ so that $\vec{F} = \vec{\nabla} \times \vec{G}$.

④ Let S^- be the part of S with the added condition that $z \leq px + qy$ for p, q constants. Given $\vec{F}$, $\vec{G}$ as above, use Stokes' theorem to evaluate

$$\iint_{S^-} \vec{F} \cdot \hat{n}\, dS$$

Some geometry problems:

(A) • Let $f(x,y,z)$ be a function so that $\vec{\nabla} f$ is 0 only at isolated points. Show that $g \vec{\nabla} f$ is not a ~~cont unless~~ gradient unless $g(x,y,z)$ is a constant.

• Let $\vec{F}(x,y,z)$ be a vector field so that $\vec{\nabla} \times \vec{F}$ is 0 only at isolated points. Show that $g \vec{\nabla} \times \vec{F}$ is not a curl unless $g(x,y,z)$ is a constant.

(B) Let $f(x,y,z)$ be a positive function so that $\vec{\nabla} f$ is 0 only at isolated points, and let ~~$f \to \infty$ as~~ f tend to infinity in all directions. (For any N, $\exists$ a compact region K so that $f|_{\mathbb{R}^3 \setminus K} \geq N$.) Then the level sets

$$S_c = \{(x,y,z) : f(x,y,z) = c\}$$

are honest compact surfaces. Let

$$v(c) = \text{volume inside } S_c$$

Prove that

$$\frac{\partial v}{\partial c} = \oint\!\!\!\oint_{S_c} \frac{\vec{\nabla} f}{\vec{\nabla} f \cdot \vec{\nabla} f} \cdot \hat{n} \, dS$$