MATH 53 14 April 08
GSI: Theo Johnson-Freyd

① (a) For what functions $f(x)$, $g(y)$ is the vector field
$$\vec{v}(x,y) = \langle f(x), g(y) \rangle$$
conservative? In terms of f, g, what is the antiderivative $F(x,y)$ so that $\vec{v} = \vec{\nabla} F$?

(b) Same question but for
$$\vec{v}(x,y) = \langle g(y), f(x) \rangle$$

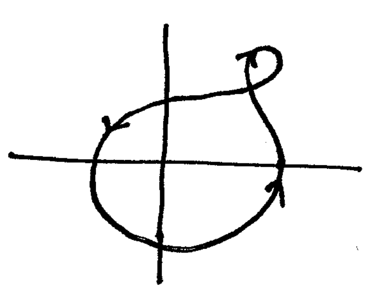
② Recall that $\vec{\nabla}(\arctan \frac{y}{x}) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$, wherever it's defined.

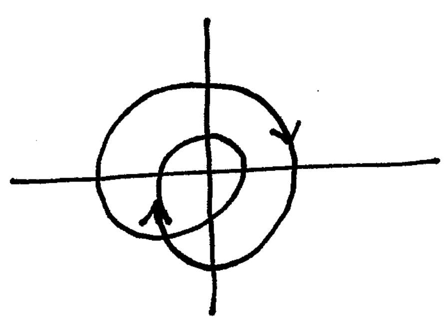
(a) In what sense is this conservative?

(b) Integrate around the circle $\vec{r}(t) = (\cos(t), \sin(t))$ $0 \le t \le 2\pi$
$$\oint \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \cdot d\vec{r}$$

(The symbol $\oint$ is used for integrals over "closed" paths, surfaces, etc., i.e. for regions that have no boundary and do not extend to infinity.)

(c) Integrate $\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ around the following loops:





(3) Newtonian mechanics is based on the differential equation

$$m\ddot{x} = P(x,y)$$
$$m\ddot{y} = Q(x,y)$$

for a vector field $\vec{v} = \langle P(x,y), Q(x,y) \rangle$. In terms of a path integral, express the difference

$$\left[\frac{m}{2}\dot{x}^2 + \frac{m}{2}\dot{y}^2 \right]_{t=0}^{1}$$

over a curve $\vec{r}(t) = (x(t), y(t))$.
(the dot marks differentiation wrt t.)

(4) Consider a complex function $f(z) = p(x+iy) + i\,q(x+iy)$.
We say f is (complex) <u>differentiable</u> if with derivative $\frac{df}{dz}$
if $$\frac{df}{dz} \overset{\text{df}}{\equiv} \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Show that $f = p + iq$ is differentiable iff the vector field $\langle p, q \rangle$ is conservative.