MATH 53 16 April 08

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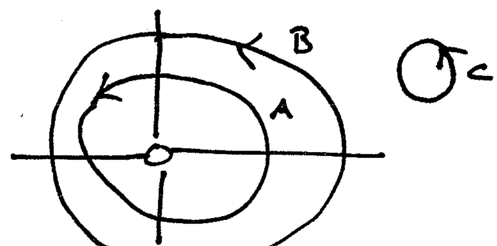
(1) Given a domain D, two loops are called <u>homologous</u> (in D) if their difference (the union of one and the reverse of the other) bounds a closed region with no other boundary. For example, in $\mathbb{R}^2 \smallsetminus \{0\}$, the curves A and B are homologous, but A and C are not:



The region may "double back" on itself. We write $A \sim B$ for "homologous".

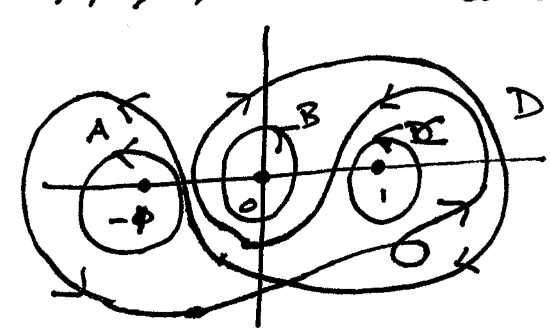
(a) Show that the relation of being homologous is an equivalence relation (if $A \sim B$ and $B \sim C$, then $A \sim C$).

(b) We can add curves by taking unions, and subtract by reversing orientation. Show that any two constant loops are homologous, and that $A + (-A) \sim$ a constant.

(c) Let $\vec{v}$ be a vector field such that $\operatorname{curl}(\vec{v}) = 0$. Show that if $A \sim B$, then:

$$\oint_A \vec{v} \cdot d\vec{r} = \oint_B \vec{v} \cdot d\vec{r}$$

(d) Let $\vec{v}$ be a vector field in $\mathbb{R}^2 \smallsetminus \{0, 1, -1\}$, and A, B, C, and D curves as shown:



If $\operatorname{curl}(\vec{v}) = 0$, and $\oint_A \vec{v} \cdot d\vec{r} = a$, $\oint_B \vec{v} \cdot d\vec{r} = b$, and $\oint_C \vec{v} \cdot d r = c$, find $\oint_D \vec{v} \cdot d\vec{r}$.

(2) Given a vector field $\vec{v} = \langle P(x,y), Q(x,y) \rangle$ in $\mathbb{R}^2$, recall that we define

$$\operatorname{curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\operatorname{div}(\vec{v}) = \vec{\nabla} \cdot \vec{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

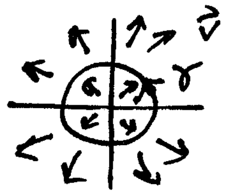
Also, define the rotation by $90°$:

$$J\vec{v} = \langle -Q(x,y), P(x,y) \rangle$$

(a) Via J, relate div and curl.

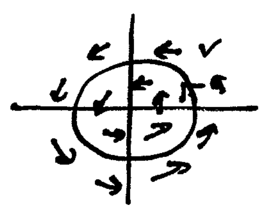
(b) The path integral $\oint_\gamma \vec{v} \cdot d\vec{r}$ measures to what extent $\vec{v}$ and the loop γ point in the same direction. Using J, define the integral, usually written $\oint_\gamma \vec{v} \times d\vec{r}$, that measures to what extent $\vec{v}$ crosses the loop γ.

The picture:



$\oint_\gamma \vec{v} \cdot d\vec{r} = 0$

$\oint_\gamma \vec{v} \times d\vec{r} > 0$



$\oint_\gamma \vec{v} \cdot d\vec{r} > 0$

$\oint_\gamma \vec{v} \times d\vec{r} = 0$

(c) If $f(x+iy) = p(x+iy) + iq(x+iy)$ is a complex function, we define the derivative, if these match, as

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$$

Show that f is differentiable iff $\vec{\nabla} \times \vec{v} = \vec{\nabla} \cdot \vec{v} = 0$ for $\vec{v} = \langle p, q \rangle$. (This was stated wrong yesterday.)

(d) State a version of Green's theorem for $\oint \vec{v} \times d\vec{r}$.