MATH 53 21 April 08
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Recall that the operator $\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \dots \rangle$, when treated as a vector, gives various differential operators:

$$\operatorname{grad} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}$$

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$$

In two dimensions, we can make the following chart:

$$\mathbb{R}^2: \quad \left\{ \begin{array}{c} \text{0-forms} \\ \shortparallel \\ \text{functions} \end{array} \right\} \xrightarrow[\text{grad}]{d} \left\{ \begin{array}{c} \text{1-forms} \\ \shortparallel \\ \text{vector fields} \end{array} \right\} \xrightarrow[\text{curl}]{d} \left\{ \begin{array}{c} \text{2-forms} \\ \shortparallel \\ \text{functions} \end{array} \right\}$$

In three dimentions, the chart is

$$\mathbb{R}^3: \quad \left\{ \begin{array}{c} \text{0-forms} \\ \shortparallel \\ \text{functions} \end{array} \right\} \xrightarrow[\vec{\nabla} = \text{grad}]{d} \left\{ \begin{array}{c} \text{1-forms} \\ \shortparallel \\ \text{v-fields} \end{array} \right\} \xrightarrow[\vec{\nabla}\times = \text{curl}]{d} \left\{ \begin{array}{c} \text{2-forms} \\ \shortparallel \\ \text{v-fields} \end{array} \right\} \xrightarrow[\vec{\nabla}\cdot = \text{div}]{d} \left\{ \begin{array}{c} \text{3-forms} \\ \shortparallel \\ \text{functions} \end{array} \right\}$$

1. Prove the "zoop-zoop" theorem, that $d^2 = 0$. I.e.,

   (a) For any function f, show $\vec{\nabla} \times (\vec{\nabla} f) = 0$

   (b) For any vector field $\vec{F}$, show that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$.

We can think of a vector field as either a 1-form or a 2-form, depending on whether we take its curl or its divergence. Conversely, if I tell you I have a j-form, you know exactly how to differentiate it. We declare that the product of a j-form and a k-form

is a $(j+k)$-form, provided $j+k \le 3 =$ dimension of space.
(Really we just declare that the only j-forms for $j<0$ or $j>3$ are ~~the 0 forms~~ zero, and all other "d"s are 0.) The "product" of two 1-forms is the cross-product; a 1-form with a 2-form is the dot product.

② Prove the Leibniz rule: $d(AB) = d(A)B + (-1)^j A\, d(B)$ if A is a j-form. i.e. prove

(a) $\vec\nabla(fg) = f\vec\nabla g + (\vec\nabla f)g$ for f, g functions

(b) $\vec\nabla\times(f\vec G) = f\,\vec\nabla\times\vec G + (\vec\nabla f)\times\vec G$ for f a fn, $\vec G$ a v-field

(c) $\vec\nabla\cdot(f\vec G) = f\,\vec\nabla\cdot\vec G + (\vec\nabla f)\cdot\vec G$

(d) $\vec\nabla\cdot(\vec F\times\vec G) = \vec F\cdot(\vec\nabla\times\vec G) - (\vec\nabla\times\vec F)\cdot\vec G$

In $\mathbb{R}^2$, Green's theorem and the F.T.C. are each a version of

$$\int_R dF = \oint_{\partial R} F$$

where F is either a 0- or a 1-form, and R is either a 1- or 2-dimensional region.

③ Be sure you understand what the boundary of a one-dimensional region is, and how to integrate over it. Your answer should make the above integral make sense.

④ Prove the two-dimensional integration by parts formula: If R is a region in $\mathbb{R}^2$ with boundary ∂R, then

$$\iint_R f(x,y)\,\vec\nabla\times\vec G(x,y)\,dx\,dy = \oint_{\partial R} f(x,y)\,\vec G(x,y)\,d\vec r - \iint_R (\vec\nabla f(x,y))\times\vec G(x,y)\,dx\,dy$$