MATH 53 28 APRIL 2008
GSI: THEO JOHNSON-FREYD

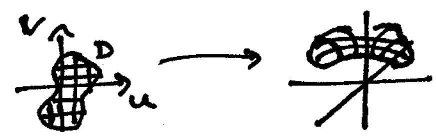
(1) Yet another 2-dimensional exercise:

Find a simple closed curve C so that the (counterclockwise) line integral is maximized:

$$\oint_C \left[(y^3-y)\,dx - 2x^3\,dy\right]$$

(2) Recall that for a parameterized surface in $\mathbb{R}^3$

$$\vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} : \quad D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$



the surface area is given by the integral

$$SA(\vec{r}(D)) = \iint_D \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\,dv$$

$$= \iint_D \sqrt{\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)^2}\, du\,dv$$

Find the analogous formula for surfaces $\begin{bmatrix} r(u,v) \\ \varphi(u,v) \\ \theta(u,v) \end{bmatrix}$ in polar coordinates.

(3) Consider a surface given by the graph of a function:

$$z = f(x,y)$$

for x, y in some domain D. Consider varying this surface by some unknown small amount

$$\Delta z = \varepsilon\, g(x,y)$$

where ε is a quantity we will let tend to 0, and $g(x,y)$ is a (smooth) function on D such that $g=0$ on the boundary.

(a) Write an expression for the surface area of the graph of the function $f(x,y)+\varepsilon g(x,y)$, and think of this as a function $S_g(\varepsilon)$.

(b) Differentiate S_g with respect to ε. Show that

$$\frac{\partial S_g(\varepsilon)}{\partial \varepsilon} = 0$$

for every g if and only if $f(x,y)$ is a harmonic function, i.e. if $\nabla^2 f = 0$.