MATH 53 31 March 08

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① For $n \geq 0$ an integer, find

$$\iiint_{\mathbb{R}^3} \sqrt{x^2+y^2+z^2}^{\,n} \, e^{-(x^2+y^2+z^2)} \, dx\, dy\, dz$$

② (a) Two-dimensional "spherical" coordinates you know:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The quarter-circle of radius $R$ is the region $0 \leq r \leq R$, $0 \leq \theta \leq \frac{\pi}{2}$. What is the area $\frac{1}{4}V_2(R)$ of the quarter circle of radius $R$? What is the derivative $\frac{1}{4}V_2'(R)$?

(b) Three-d spherical coordinates:

$$x = r\cos\theta_1 \cos\theta_2$$
$$y = r\sin\theta_1 \cos\theta_2$$
$$z = r \sin\theta_2$$

The eighth-sphere is $0 \leq r \leq R$, $0 \leq \theta_1, \theta_2 \leq \frac{\pi}{2}$. What is the volume $\frac{1}{8}V_3(R)$ of the eighth-sphere? What is the derivative $\frac{1}{8}V_3'(R)$?

(c) In four-d, we have

$$x = r\cos\theta_1 \cos\theta_2 \cos\theta_3$$
$$y = r\sin\theta_1 \cos\theta_2 \cos\theta_3$$
$$z = r \sin\theta_2 \cos\theta_3$$
$$w = r \sin\theta_3$$

What is the (4-d) volume $V_4(R)$ of ~~the~~ one sixteenth of four-dimensional sphere ($0 \le r \le R$, $0 \le \theta_1, \theta_2, \theta_3 \le \frac{\pi}{2}$)?

To find this, you will need to know the volume form

$$dV = dx\,dy\,dz\,dw$$

in terms of the spherical coordinates. Hint: use the change-of-variables formula from §12.8.

(d) Can you guess the formula for the volume of the unit $k$-dimensional sphere? Hint: ~~what is~~ the pattern is easiest if you look at just evens or just odds.

③ Here's another way to find the volume of the $k$-sphere. We know it must be $V_k(R) = R^k \cdot v_k$ for some number $v_k$. The unit $k$-sphere is all the points $x_1, \ldots, x_k$ so that $x_1^2 + x_2^2 + \ldots + x_k^2 \le 1$.

Think of this as

$$\begin{cases} x_1^2 + x_2^2 \le 1 \\ x_3^2 + \ldots + x_k^2 \le 1 - (x_1^2 + x_2^2) \end{cases}$$

Then set up an integral for the volume of the $k$-sphere as an integral over the unit circle of the volume of the $(k-2)$-sphere, and evaluate the integral.