MATH 53 3 March 08

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① (a) On the quiz last time, we considered the ellipsoid

$$5 = 2x^2 + 2y^2 + z^2 + 2xy - 2xz$$

which defines $z(x,y)$ implicitly, as a multi-valued function. Show that $z(x,y)$ is single-valued at a point (x,y) if and only if $\frac{\partial z}{\partial x} = \infty$ at that point. Conclude that $\frac{\partial z}{\partial x} = \infty$ if and only if $\frac{\partial z}{\partial y} = \infty$ (i.e. they are ∞ at the same points (x,y).).

(b) Interpret this geometrically, and generalize to any quadratic surface (a surface is <u>quadratic</u> if it is the level set of some $f(x,y,z)$ a polynomial of total degree 2).

② (a) Given a function $z = f(x,y)$, we can define the tangent plane at x_0, y_0. Write this tangent plane as a function $z = g(\Delta x, \Delta y)$, where $\Delta x = x - x_0$, $\Delta y = y - y_0$. For $\Delta x, \Delta y$ small, we have $f(x,y) \approx g(\Delta x, \Delta y)$; how large, roughly, is the difference?

(b) If we forget about y, and think of $f(x,y)$ as a function just in x, we can apply Taylor's theorem (from Math 1B). Write out the result — the coefficients will be functions in y.

(c) Now apply Taylor's theorem to each coefficient, and write $f(x_0+\Delta x, y_0+\Delta y)$ as a double power series in Δx and Δy.

(d) If you're comfortable with vectors, write your answers to (a) and (c) in terms of vectors and matrices.

(e) Understand (a) as the first term in a Taylor expansion.

(f) Generalize your results to functions of arbitrarily many variables. Even better: What if f is a function of <u>infinitely many</u> variables?