MATH 53 4 April 08
GSI: Theo Johnson-Freyd

(1)(a) Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be constant vectors, and let $\vec{r} = \langle x, y, z \rangle$ be the position vector. Let E be the region $0 \le \vec{a}\cdot\vec{r} \le \alpha$, $0 \le \vec{b}\cdot\vec{r} \le \beta$, $0 \le \vec{c}\cdot\vec{r} \le \gamma$, where α, β, and γ are positive constants. Find

$$\iiint_E (\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})(\vec{c}\cdot\vec{r})\, dV$$

(b) In terms of an integral over the unit cube, for an unknown function $f(\vec{r})$, find

$$\iiint_E f(\vec{r})\, dV$$

(2) Spherical coordinates in n dimensions

Consider the following change of variables

$$x_1 = r\cos\theta_1$$
$$x_2 = r\sin\theta_1\cos\theta_2$$
$$x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3$$
$$x_4 = r\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4$$
$$\vdots$$
$$x_{n-1} = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1}$$
$$x_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}$$

from coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ to $(x_1, \ldots, x_n)$.

(a) When $n = 3$, calculate the Jacobian $\left|\frac{\partial(x_1, x_2, x_3)}{\partial(r, \theta_1, \theta_2)}\right|$.

(b) How about when $n=4$? Hint: expand by minors along either the first row or the last column; write in terms of the answer to (a).

(c) Find a formula for the Jacobian $\left|\frac{\partial(x_1, \ldots, x_n)}{\partial(r, \theta_1, \ldots, \theta_{n-1})}\right|$.

Hint: Expand by minors along the last column, and use induction.

③ (a) Sketch the vector fields in $\mathbb{R}^2$:

- $\hat{i} + \hat{j}$
- $x\hat{i} + y\hat{j}$
- $y\hat{i} - x\hat{j}$
- $\vec{\nabla}\left(\sqrt{x^2+y^2}\right)$
- $\vec{\nabla}\left(\arctan\left(\frac{y}{x}\right)\right)$

(b) N-dimensional gravity is given by the potential

$$f(\vec{r}) = \frac{G}{|\vec{r}|^{n-2}}$$

where G is a constant and $n \geq 3$. Sketch the vector field $\vec{\nabla} f(\vec{r})$.

④ Describe how to use ~~a vec~~ the flow lines of a vector field to solve the differential equation

$$\frac{dy}{dx} = f(x, y).$$

I.e. what vector field should you use?