MATH 53 5 MARCH 08

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① Let $f(x,y,z) = x^2 + y^3 + z^4$, and consider the parameterized curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$
$= \langle \sin(t), t, e^t \rangle$. ~~Define $g(t) = f(\vec{r}(t))$.~~

(a) Find $\frac{\partial f}{\partial t}$ at $t=0$ by substituting in, and evaluating directly.

(b) Find $\vec{\nabla} f$ at $\langle 0,0,1 \rangle$, and find $\frac{\partial \vec{r}}{\partial t}$ at $t=0$.

(c) Interpret the chain rule

$$\vec{\nabla} f \cdot \frac{\partial \vec{r}}{\partial t} = \frac{\partial f}{\partial t}$$

geometrically; remember that the left-hand side is a directional derivative.

② Let $\vec{h}(s)$ be a parameterized curve that stays in a level surface $f(x,y,z) = 1$. What can you say about $\frac{\partial f}{\partial s}$? Thus, what is the geometric relationship between $\frac{\partial \vec{h}}{\partial s}$ and $\vec{\nabla} f$?

③ Chain rule with matrices:

If you are comfortable with matrices,

Let $f_1(x_1,x_2,x_3)$ and $f_2(x_1,x_2,x_3)$ be two functions each of three variables: we can write this as a vector function $\vec{f}(\vec{x})$ where $\vec{f} = \langle f_1, f_2 \rangle$ and $\vec{x} = \langle x_1, x_2, x_3 \rangle$, and let each x_i be a function of two variables t_1, t_2, so that ~~f~~ $\vec{t} = \langle t_1, t_2 \rangle$, we have a vector function $\vec{x}(\vec{t})$. Define the derivative matrix

$$\frac{\partial \vec{f}}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{pmatrix}$$

and a similar 3×2 matrix for $\frac{\partial \vec{x}}{\partial \vec{t}}$. Use the chain rule to write the 2×2 matrix $\frac{\partial \vec{f}}{\partial \vec{t}}$ in terms of the 2×3 matrix $\frac{\partial \vec{f}}{\partial \vec{x}}$ and the 3×2 matrix $\frac{\partial \vec{x}}{\partial \vec{t}}$.

4) From 1A, we know that maxima and minima of functions occur at points where the derivative is zero; if $f(x,y,z)$ is a function of three variables, extrema occur only if all partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ are zero. (a) What are possible extrema of

$$f(x,y,z) = \sin x + yz \, ?$$

(b) What are the maxima along the curve

$$x(t) = t^2, \quad y(t) = \cos t, \quad z(t) = e^t \quad ?$$

Interpret your use of the chain rule geometrically.

(c) What about on the surface $z = x+y$?

Again, you'll use the chain rule; interpret geometrically.