MATH 53 5 MAY 08
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① (a) If a vector field $\vec{E}$ with $div(\vec{E}) = 0$ is defined everywhere except at the origin, what can you say about the integral of $\vec{E}$ over various spheres in $\mathbb{R}^3$?

(b) Let $S(r)$ be the sphere of radius r centered at the origin in $\mathbb{R}^3$, and say that

$$\oint\!\!\oint_{S(1)} \vec{E} \cdot d\vec{S} = 1$$

And let's assume that $div(\vec{E}) = 0$ and that $\vec{E}$ is spherically symmetric: i.e. that $\vec{E}(x,y,z)$ points towards or away from the origin, and depends only on $\sqrt{x^2+y^2+z^2}$: in physics talk, we'd write $\vec{E} = E(r)\,\hat{r}$. Then find $\vec{E}(x,y,z)$.

(c) What is $curl(\vec{E})$? Can you find a function $P(x,y,z)$ so that $\vec{E} = \vec{\nabla}(P)$?

(d) For your answer in (b), can you define $div(\vec{E})$ at the origin? Let $\delta(x)$ be the "function" so that $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$ for any function $f : \mathbb{R} \to \mathbb{R}$.

Then "$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$". Write $div(\vec{E})$ in terms of $\delta(x)$, $\delta(y)$, and $\delta(z)$.

(e) Show that $f(y) = \int_{-\infty}^{\infty} f(x)\,\delta(x-y)\,dx$. Hence, given $\rho(x,y,z)$, a function on $\mathbb{R}^3$ with "compact support", use ~~the~~ your answer in (c) to find a function $Q(x,y,z)$ so that $\nabla^2 Q = \rho$. It turns out that this is the unique function satisfying this equation that "dies quickly" at infinity.

(2) Let $\vec{B}$ be a vector field with $\mathrm{div}(\vec{B}) = 0$ and $\mathrm{curl}(\vec{B}) = \delta(x)\,\delta(y)$.

(a) Given any loop S, find $\oint_S \vec{B} \cdot d\vec{r}$.

(b) If $\vec{B}$ is cylindrically symmetric — i.e. $\vec{B}(x,y,z)$ ~~draws~~ points parallel to $y\hat{\imath} - x\hat{\jmath}$, and the magnitude depends only on $\sqrt{x^2+y^2}$ — in physics notation: $\vec{B} = B(\sqrt{x^2+y^2})\,\hat{\varphi}$ — then find $\vec{B}(x,y,z)$.

(c) Let $\vec{J}$ be any vector field with $\mathrm{div}(\vec{J}) = 0$. Find a $\vec{B}$ so that $\mathrm{curl}(\vec{B}) = \vec{J}$.

(d) Hence find a vector field $\vec{A}$ so that $\mathrm{curl}^2(\vec{A}) = \mathrm{curl}(\mathrm{curl}(\vec{A})) = \vec{J}$.

ALL together, you've <u>solved</u> Electromagnetic statics:

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} = \vec{J}$$

and found "potentials" Q, $\vec{A}$ so that $\vec{E} = \vec{\nabla} Q$, $\vec{B} = \vec{\nabla} \times \vec{A}$.