MATH 53 7 April 08
GSI: Theo Johnson-Freyd

(1) (a) Sketch the vector fields in $\mathbb{R}^2$:

- $\hat{\imath} + \hat{\jmath}$
- $x\hat{\imath} + y\hat{\jmath}$
- $y\hat{\imath} - x\hat{\jmath}$
- $\vec{\nabla}\left(\sqrt{x^2+y^2}\right)$
- $\vec{\nabla}\left(\arctan(y/x)\right)$

(b) n-dimensional (Newtonian) gravity is given by the potential energy function

$$f_n(\vec{r}) = \frac{G_n}{|\vec{r}|^{n-2}}$$

where G_n is a constant an $n \geq 3$. Sketch the vector field $\vec{\nabla} f(\vec{r})$.

(When $n=2$, $f_2(\vec{r}) = G_2 \ln|\vec{r}|$, and when $n=1$, $f_1(\vec{r}) = G_1|\vec{r}|$. Also sketch the corresponding vector fields in dimensions 1 and 2.)

(2) Describe how to use the flow lines of a vector field to solve the differential equation

$$\frac{dy}{dx} = f(x,y)$$

What vector field should you use?

③ Let's say we have a radial path in $\mathbb{R}^2$ given by

$$\vec{r}(t) = (a+t, 0)$$

where $a>0$ is a constant, and t ranges from $0 \le t \le 1$.

(a) For each of the vector fields $\vec{v}(\vec{r})$ in ①(a), calculate

$$\int_{t=0}^{1} \vec{v}(\vec{r}(t)) \cdot \frac{\partial \vec{r}}{\partial t}\, dt = \int_{\text{path}} \vec{v}(\vec{r})\, d\vec{r}$$

(b) Let $F(\vec{r}) = f(|\vec{r}|)$ depend only on the radius $|\vec{r}|$.
What is $\vec{\nabla} F$? What is $\int_{t=0}^{1} \vec{\nabla} F(\vec{r}(t)) \cdot \frac{\partial \vec{r}}{\partial t}\, dt$?
Answer in terms of f.

(c) Let $\vec{v}(\vec{r})$ be centrally symmetric — i.e. $\vec{v}(\vec{r}) = g(|\vec{r}|) \cdot \vec{r}$.
In terms of g, what is

$$\int_{t=0}^{1} \vec{v}(\vec{r}(t)) \cdot \frac{\partial \vec{r}}{\partial t}\, dt$$

for $\vec{r}$ as above?

(d) Let $\vec{v}(\vec{r})$ continue to be centrally symmetric, and consider the path

$$\vec{s}(t) = (t, 0)$$

for t ranging over the positive reals. Define

$$G(T) = \int_{t=1}^{T} \vec{v}(\vec{r}(t)) \cdot \frac{\partial \vec{s}}{\partial t}\, dt$$

What is $G'(T)$? Answer in terms of g (defined in (c)).